An Alexandrov-Bakelman-Pucci estimate for an anisotropic Laplacian with positive drift in unbounded domains

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Abstract

We show an Alexandrov-Bakelman-Pucci (ABP) estimate for a PDE with anisotropic Laplacian in two dimensions in unbounded domains, where the drift vector varies in a segment of the positive quadrant . The unbounded domains are assumed to be bounded below in the x-direction, as well as in the y-direction. The constant in the upper estimate of the ABP-estimate, which depends in the usual theorems for bounded domains on the diameter of this domain, depends in our case on the small parameter ϵ, appearing in the anisotropic Laplacian. The result is motivated by certain problems of singular perturbation in stochastic control theory, and our methods are probabilistic.

1 Introduction

Let us start with the motivation for this paper. In singular perturbation theory one considers boundary value problems, where the differential operator depends on a small parameter, say ϵ. One usually constructs a so called formal approximation for the solution of the problem, using techniques as matched asymptotic expansion (see e.g. [5] or [16]). Plugging this formal approximation in the PDE, yields a residuum, which is small in a certain norm. The aim is then to show that the formal approximation is indeed close to the full solution of the problem. A special case of these problems are equations where an anisotropic Laplacian appears.

The application we have in mind comes from stochastic control theory. Restricting to the two-dimensional situation, the problem is the following. The wealth of two companies is given by a two-dimensional Brownian motion, where the volatility of the second company is small in comparison to the first one. Moreover, a controller (in the model of [15] this is the government by an appropriate tax policy) can influence the drift of the companies in a way, s.t. the total drift adds up to a certain positive constant. Let us remark that it is mathematically equivalent to consider two companies

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supporting each other by transfer payments. The goal is to maximize the probability that both companies survive.

Now, in [9] we were able to construct a formal solution of the corresponding Hamilton-Jacobi-Bellman equation with the methods of singular perturbation theory, s.t. the difference of the full solution and the formal approximation, say $w(x, y)$, fulfills a boundary value problem on the positive quadrant, involving an anisotropic Laplacian and with a residual inhomogeneity, say $f(x, y)$, which has an L^p-norm, $p \in [1, \infty)$, of order $O(\epsilon^2)$. In order to be sure that our approximation is a proper one in the whole positive quadrant, we would need a result, which ensures a small L^∞-norm of w. Such results are usually provided by so called Alexandrov-Bakelman-Pucci (ABP) estimates.

Indeed, ABP-estimates give an upper bound for the solution of a certain boundary value problem in the L^∞-norm in terms of the L^p norm of the inhomogeneity of the PDE (assuming homogeneous boundary conditions, which we shall do). Now, for problems with bounded domains, say G, the problem is well understood, see e.g. [7], and it is known that the constant, say C, in the upper estimate $C||f||_{L^p}$ depends on the diameter of the domain.

For unbounded domains the situation is more complicated. E.g., even restricting to domains which are bounded, say below, in one direction the weak maximum principle (MP) - and hence an ABP-estimate - is not necessarily valid. Indeed, consider

$$\begin{aligned} w_x + \frac{1}{2}\Delta w &= 0, \\ w(0, y) &= 0, \;\; y \in \mathbf{R}, \\ G &= \mathbf{R}^+ \times \mathbf{R}, \end{aligned}$$

which has the solution $w = 1 - e^{-2x}$. Restricting further to domains bounded below in both directions is still not sufficient to guarantee the MP, as the example from [15] shows: The function $w(x, y) = 1 - e^{-2\min(x,y)} - 2\min(x, y)e^{-x-y}$ is a C^2-solution of the system

$$\begin{aligned} w_x \mathbf{1}_{\{y>x\}} + w_y \mathbf{1}_{\{y \le x\}} + \frac{1}{2}\Delta w &= 0, \\ G &= \mathbf{R}^+ \times \mathbf{R}^+, \\ w_{/\partial G} &= 0. \end{aligned}$$

Hence, we need a further restriction, and we choose the boundary behavior of our function $w(x, y)$ for $x \to \infty, y \to \infty$, as this is motivated by our application.

Summing up, we shall consider problems on $G \subset \mathbf{R}^+ \times \mathbf{R}^+$, with anisotropic Laplacian and a drift vector, varying in a segment of the positive quadrant, as well as a “boundary condition” at (∞, ∞).

Concerning the existing literature, let us mention the paper of [2], where the author shows an ABP-estimate for unbounded domains, satisfying a condition called (G), which, roughly speaking, assumes “enough boundary near every point of the domain”; see Remark 2.1 of [2] for some examples satisfying this condition. Cones like $\mathbf{R}^+ \times \mathbf{R}^+$ do *not* satisfy his condition.

A. Vitolo weakens condition (G) in [18] to a condition (wG), s.t., e.g., cones *do* fulfill condition (wG). But on the other hand, he has to assume a decrease of the order $O(1/x)$ for the norm of the drift vector, as $||x|| \to \infty$, whereas in [2] only boundedness of the drift vector is assumed.

2 The main result

Let us start with some notation. Let G be an unbounded domain in $\mathbf{R}^2$, satisfying and exterior cone condition, see e.g. [7]. Moreover, we assume that the domain G is bounded below in the x-direction, as well as in the y-direction, i.e. w.l.o.g. we assume

$$G \subset \mathbf{R}^+ \times \mathbf{R}^+. \tag{1}$$

Furthermore, we introduce an anisotropic Laplacian by $\Delta^{(\epsilon)} := \frac{\partial^2}{\partial x^2} + \epsilon^2 \frac{\partial^2}{\partial y^2}$, where $\epsilon > 0$ is assumed to be small. We shall study the following system

$$\begin{aligned} \mathcal{L}w + f := a_1(x,y)w_x(x,y) + a_2(x,y)w_y(x,y) + \frac{1}{2}\Delta^{(\epsilon)}w(x,y) + f(x,y) = 0, \\ w_{/\partial G} = 0, \\ \lim_{x\to\infty, y\to\infty} w(x,y) = 0. \end{aligned} \tag{2}$$

For the drift vector we assume $ka_1 + a_2 = \rho$, with constants $k, \rho > 0$, $a_i \geq 0$ (hence bounded) and measurable, which means the the drift vector may vary on a line segment of the positive quadrant. In the application, mentioned in the Introduction, one would have $k = \rho = 1$. We are interested in the case of small ϵ, hence we shall provide the explicit dependence of constants on ϵ, but we put the dependence on k and ρ in generic positive constants, which may vary from place to place, i.e., we shall write $C(k,\rho)$ or simply C.

For the inhomogeneity $f(x,y)$ we assume $f \in L^1(G) \cap L^2(G) \cap L^\infty(G)$. Our main result is

Theorem 2.1 *The system (2) has a unique solution $w \in W^{2,2}_{loc}(G) \cap C(G)$, which fulfills*

$$||w||_{L^\infty(G)} \leq \frac{C(\rho,k)}{\epsilon}\left(\sqrt{\epsilon}(-\ln\epsilon)||f||_{L^2(G)} + ||f||_{L^1(G)}\right).$$

We shall prove the theorem by a series of auxiliary results and start with the definition of the system

$$\begin{aligned} \mathcal{L}w^{(R)} + f(x,y) = 0, \\ w^{(R)}_{/\partial G^{(R)}} = 0, \end{aligned} \tag{3}$$

with $G^{(R)} := G \cap (0,R)^2$, and where the a_i and the inhomogeneity f stem from (2) (restricted to the set $G^{(R)}$).

For this system we find

Lemma 2.1 *The system (3) has a unique solution $w^{(R)} \in W^{2,2}_{loc}(G^{(R)}) \cap C(\overline{G^{(R)}})$, which allows the representation*

$$w^{(R)}(x,y) = \mathbf{E}\left[\int_0^{\tau^{(R)}} f(Z_t)\,dt\right],$$

where $\tau^{(R)} := \inf\{t > 0 | Z_t \notin G^{(R)}\}$, with $Z_t := (X_t, Y_t)$,

$$\begin{aligned} X_t &= x + \int_0^t a_1(X_s,Y_s)\,ds + W^{(1)}_t, \\ Y_t &= y + \int_0^t a_2(X_s,Y_s)\,ds + \epsilon W^{(2)}_t, \end{aligned}$$

and $W^{(1)}, W^{(2)}$ independent standard Brownian motions.

Proof. We first note that $G^{(R)}$ satisfies an exterior cone condition, since G does so. Hence, we can apply Theorem 9.30 of [7], to get a unique solution $w^{(R)} \in W^{2,2}_{loc}(G^{(R)}) \cap C(\overline{G^{(R)}})$.

On the other hand, Theorem 2.10.2 of [13] gives the desired representation, if we apply the Theorem there once for v, and once for $-v$ (in the notation there!). □

Remark 2.1 *For convenience we assume $f \geq 0$, prove Theorem 2.1 first in this case and close the proof with the general case.*

Moreover, we define $f(x,y)$ in the complement of G to be zero. Slightly abusing notation, we denote the function defined now on $\mathbf{R}^2$ again by f.

Defining now

$$W(x,y) := \mathbf{E}\left[\int_0^\infty f(Z_t)\,dt\right], \tag{4}$$

we get as an easy consequence of our representation formula and the nonnegativity of f

Corollary 2.1

$$w^{(R_1)}(x,y) \le w^{(R_2)}(x,y) \le W(x,y),$$

for $(x,y) \in G^{(R_1)}$, *and for* $R_1 \le R_2$.

Our next lemma gives an upper bound for the function W, by replacing the function f by an appropriate simple function, i.e. we have

Lemma 2.2 *Let* $\mathbf{R}^2 = \cup_{i\in\mathbb{Z}^2} Q_i$, *with*

$$Q_i := [x+(2j-1)\Delta, x+(2j+1)\Delta) \times [y+(2k-1)\Delta, y+(2k+1)\Delta).$$

Here $i=(j,k)$, *with* $j,k \in \mathbb{Z}$ *and* $\Delta > 0$. *Finally, let* $\overline{f}_i := ess\ sup_{(x,y)\in Q_i} f(x,y)$. *Then we have*

$$W(x,y) \le \sum_{i\in\mathbb{Z}^2} \overline{f}_i \mathbf{E}\left[\int_0^\infty \mathbf{1}_{Q_i}(X_s,Y_s)\,ds\right],$$

where $\mathbf{1}$ *denotes the indicator function.*

Proof.

$$\mathbf{E}\left[\int_0^\infty f(X_s,Y_s)\,ds\right] \le \mathbf{E}\left[\int_0^\infty \sum_{i\in\mathbb{Z}^2} \overline{f}_i \mathbf{1}_{Q_i}(X_s,Y_s)\,ds\right] = \sum_{i\in\mathbb{Z}^2} \overline{f}_i \mathbf{E}\left[\int_0^\infty \mathbf{1}_{Q_i}(X_s,Y_s)\,ds\right] \quad \square$$

In order to get a more convenient process in one direction, we now introduce new processes:

$$\begin{aligned} \overline{X}_t &:= (kX_t + Y_t)/\sqrt{\epsilon^2+k^2} = \overline{x} + \overline{\rho}t + B_t^{(1)}, \\ \overline{Y}_t &:= (-\epsilon X_t + kY_t/\epsilon)/\sqrt{\epsilon^2+k^2} = \overline{y} + \int_0^t \overline{v}(\overline{X}_s,\overline{Y}_s)\,ds + B_t^{(2)}, \end{aligned} \tag{5}$$

where $B_t^{(1)}$ and $B_t^{(2)}$ are *independent* standard Brownian motions, and where

$$\begin{aligned} \overline{\rho} &= \rho/\sqrt{\epsilon^2+k^2}, \\ \overline{v}(\overline{X}_s,\overline{Y}_s) &\in \left[\frac{-\overline{\rho}\epsilon}{k}, \frac{\overline{\rho}k}{\epsilon}\right], \end{aligned}$$

holds. This can be verified by direct calculation. Our next lemma estimates the time spend in the small squares (we first consider the one around the origin) appearing in the previous lemma, by our new processes. I.e., we have

Lemma 2.3

$$\begin{aligned} J(x,y,a_1) &:= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{[-\Delta,\Delta)\times[-\Delta,\Delta)}(X_s,Y_s)\,ds\right] \\ &\le \int_0^\infty \mathbf{P}\left(-C(k)\Delta \le \overline{X}_t < C(k)\Delta, -\frac{C(k)\Delta}{\epsilon} \le \overline{Y}_t < \frac{C(k)\Delta}{\epsilon}\right) dt, \end{aligned}$$

for some positive constant $C(k)$.

Proof. Using the "backwards transformation"

$$\begin{aligned} X_t &= \frac{k}{\sqrt{\epsilon^2+k^2}}\left(\overline{X}_t - \frac{\epsilon}{k}\overline{Y}_t\right), \\ Y_t &= \frac{1}{\sqrt{\epsilon^2+k^2}}\left(\epsilon^2\overline{X}_t + \epsilon k\overline{Y}_t\right), \end{aligned}$$

we express our target functional by our new variables and get

$$\begin{aligned} J(\overline{x},\overline{y},\overline{v}) &= \int_0^\infty \mathbf{P}\left(-\Delta \le \frac{k\overline{X}_t - \epsilon\overline{Y}_t}{\sqrt{\epsilon^2+k^2}} < \Delta, -\frac{\Delta}{\epsilon} \le \frac{\epsilon\overline{X}_t + k\overline{Y}_t}{\sqrt{\epsilon^2+k^2}} < \frac{\Delta}{\epsilon}\right) dt \\ &= \int_0^\infty \mathbf{P}\left(\frac{-\Delta\sqrt{\epsilon^2+k^2}+k\overline{X}_t}{\epsilon} < \overline{Y}_t \le \frac{\Delta\sqrt{\epsilon^2+k^2}+k\overline{X}_t}{\epsilon},\right. \\ &\qquad \left. -\frac{\Delta\sqrt{\epsilon^2+k^2}}{\epsilon k} - \frac{\epsilon}{k}\overline{X}_t \le \overline{Y}_t < \frac{\Delta\sqrt{\epsilon^2+k^2}}{\epsilon k} - \frac{\epsilon}{k}\overline{X}_t\right) dt \\ &\le \int_0^\infty \mathbf{P}\left(-C(k)\Delta \le \overline{X}_t < C(k)\Delta, -\frac{C(k)}{\epsilon}\Delta \le \overline{Y}_t < \frac{C(k)}{\epsilon}\Delta\right) dt, \end{aligned}$$

where the last inequality follows by simple geometric considerations. □

We restrict now to the time interval $[0,1]$ and formulate a proposition, which we shall prove in the next section, mainly by considerations about the one-dimensional process in y-direction. In order to do this, we have to introduce some new processes, which are just a shift in the y-direction.

$$\begin{aligned} X_t^{(1)} &= x^{(1)} + \rho^{(1)}t + B_t^{(1)}, \\ Y_t^{(1)} &= y^{(1)} + \int_0^t v_s^{(1)}\, ds + B_t^{(2)}, \\ \rho^{(1)} &:= \frac{\rho}{\sqrt{k^2+\epsilon^2}} = \overline{\rho}, \\ v^{(1)}(X_t^{(1)}, Y_t^{(1)}) &\in [0,M], M > \max(\rho^{(1)}, 2). \end{aligned}$$

Note that as we are finally interested in small values of ϵ, which provides large values of M, we assume - for convenience - $M > \max(\rho^{(1)}, 2)$. For this processes we get

Proposition 2.1 *One has*

$$\begin{aligned} a)\, \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_s^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{(1)})\, ds\right] &\le C(\rho,k)\Delta\tilde{\Delta}\left(\left|\ln\left(\max(|x^{(1)}|,|y^{(1)}|)\wedge 1\right)\right| + \ln M\right), \\ b)\, \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_s^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{(1)})\, ds\right] &\le C(\rho,k)\Delta\tilde{\Delta}e^{-\frac{(x^{(1)})^2}{32}}(\ln M),\ |x^{(1)}| > 4\rho^{(1)}, \\ c)\, \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_s^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{(1)})\, ds\right] &\le C(\rho,k)\Delta\tilde{\Delta}e^{-\frac{(y^{(1)})^2}{32}}(\ln M),\ |y^{(1)}| > 4M, \\ d)\, \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_s^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{(1)})\, ds\right] &\le C(\rho,k)\Delta\tilde{\Delta}e^{-\frac{(x^{(1)})^2+(y^{(1)})^2}{32}}(\ln M), \end{aligned}$$

for $|x^{(1)}| > 4\rho^{(1)}, |y^{(1)}| > 4M$.

We now go back to the processes $(\overline{X}_t, \overline{Y}_t)$ and show a similar result as the previous one by an application of Girsanov's theorem. In the rest of this section we shall use $\tilde{\Delta} := \Delta/\epsilon$

Corollary 2.2

$$\begin{aligned} J^{(1)}(\overline{x},\overline{y},\overline{v}) &:= \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_s)\, ds\right] \\ &\le C(\rho,k)\Delta\tilde{\Delta}\left(|\ln(\max(|\overline{x}|,|\overline{y}|)\wedge 1)| - \ln\epsilon\right), \\ J^{(1)}(\overline{x},\overline{y},\overline{v}) &\le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)e^{-\frac{(\overline{x})^2}{32}},\ |\overline{x}| > 4\overline{\rho}. \end{aligned}$$

Proof. We first note that, if we define $\tilde{B}_t := B_t^{(2)} - \frac{\overline{\rho}\epsilon}{k}t$, $(B_t^{(1)}, \tilde{B}_t)$ is by Girsanov's theorem a two-dimensional $\mathbf{Q}$-Brownian motion, where $Z_t = \frac{d\mathbf{Q}}{d\mathbf{P}}$, with $1/Z_t = e^{-\frac{\overline{\rho}\epsilon}{k}\tilde{B}_t - \frac{\overline{\rho}^2\epsilon^2}{2k^2}t}$. Moreover, we have $v^{(1)} = \overline{v} + \frac{\overline{\rho}\epsilon}{k} \in [0, \frac{\overline{\rho}k}{\epsilon} + \frac{\overline{\rho}\epsilon}{k} =: M]$, and $\overline{x} = x^{(1)}, \overline{y} = y^{(1)}$. Hence,

$$\begin{aligned}
& \int_0^1 \mathbf{P}\left(-\Delta \leq \overline{x} + \overline{\rho}t + B_t^{(1)} < \Delta, -\tilde{\Delta} \leq \overline{y} + \int_0^t \overline{v}_s\, ds + B_t^{(2)} < \tilde{\Delta}\right) dt \\
= & \int_0^1 \mathbf{P}\left(-\Delta \leq \overline{x} + \overline{\rho}t + B_t^{(1)} < \Delta, -\tilde{\Delta} \leq \overline{y} + \int_0^t v_s^{(1)}\, ds + \tilde{B}_t < \tilde{\Delta}\right) dt \\
= & \int_0^1 \mathbf{E_Q}\left[\frac{1}{Z_t}\mathbf{1}_{\{-\Delta \leq \overline{x}+\overline{\rho}t+B_t^{(1)}<\Delta, -\tilde{\Delta}\leq \overline{y}+\int_0^t v_s^{(1)}\, ds+\tilde{B}_t<\tilde{\Delta}\}}\right] dt \\
= & \int_0^1 \mathbf{E_Q}\left[e^{-\frac{\overline{\rho}\epsilon}{k}\tilde{B}_t - \frac{\overline{\rho}^2\epsilon^2}{2k^2}t}\ \mathbf{1}_{\{-\Delta \leq \overline{x}+\overline{\rho}t+B_t^{(1)}<\Delta, -\tilde{\Delta}\leq \overline{y}+\int_0^t v_s^{(1)}\, ds+\tilde{B}_t<\tilde{\Delta}\}}\right] dt \\
\leq & \ C(\rho,k)\int_0^1 \mathbf{E_Q}\left[\mathbf{1}_{\{-\Delta \leq \overline{x}+\overline{\rho}t+B_t^{(1)}<\Delta, -\tilde{\Delta}\leq \overline{y}+\int_0^t v_s^{(1)}\, ds+\tilde{B}_t<\tilde{\Delta}\}}\right] e^{\frac{\overline{y\rho}\epsilon}{k}}\, dt,
\end{aligned}$$

where we have used in the last inequality the fact that

$$-\frac{\overline{\rho}\epsilon}{k}\tilde{B}_t \leq \frac{2M\overline{\rho}\epsilon}{k} + \frac{\overline{y\rho}\epsilon}{k}$$

holds on the set where the indicator function is nonzero, for Δ small enough. Applying Proposition 2.1, in particular c) to get control over the $e^{\frac{\overline{y\rho}\epsilon}{k}}$ term, concludes the proof. □

We extend this result to the full time interval and assert the validity of

Lemma 2.4

$$\begin{aligned}
J^{(\infty)}(\overline{x},\overline{y},\overline{v}) &:= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_s)\, ds\right] \\
&\leq C(\rho,k)\Delta\tilde{\Delta}\left(|\ln(\max(|\overline{x}|,|\overline{y}|)\wedge 1)| - \ln\epsilon\right).
\end{aligned}$$

Proof. Since for $\int_0^1$ we have already the previous corollary, we concentrate on the rest of the time interval and get

$$\mathbf{E}\left[\int_1^\infty \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_s)\, ds\right] = \sum_{n=1}^\infty \mathbf{E}\left[\int_n^{n+1} \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_s)\, ds\right]. \tag{6}$$

Fixing now n, we define

$$\begin{aligned}
& \mathbf{E}\left[\int_n^{n+1} \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_t)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_t)\, dt\right] \\
= & \int_{-\infty}^\infty \mathbf{P}\left(\overline{X}_n \in dz\right)\mathbf{E}\left[\int_n^{n+1} \mathbf{1}_{[-\Delta,\Delta)}(\overline{X}_t)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\overline{Y}_t)\, dt | \overline{X}_n = z\right] \\
= & \int_{|z|>4\overline{\rho}} + \int_{|z|\leq 4\overline{\rho}} =: J_1(n) + J_2(n).
\end{aligned} \tag{7}$$

For $J_1(n)$ we have by Corollary 2.2

$$\begin{aligned}
J_1(n) &\leq C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)\int_{|z|>4\overline{\rho}} \mathbf{P}\left(\overline{X}_n \in dz\right) e^{-\frac{z^2}{32}} \\
&= C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)\int_{|z|>4\overline{\rho}} \frac{1}{\sqrt{2\pi n}} e^{-\frac{(z-\overline{x}-n\overline{\rho})^2}{2n}} e^{-\frac{z^2}{32}}\, dz \leq \frac{C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)}{\sqrt{n}} e^{-\frac{(n\overline{\rho}+\overline{x})^2}{2n+32}},
\end{aligned}$$

where the last inequality holds by a trivial estimate of the error functions, resulting from the integral. Hence, we get for the sum

$$\begin{aligned}
\sum_{n=1}^\infty J_1(n) &\leq C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{-\frac{(n\overline{\rho}+\overline{x})^2}{2n+32}} \leq C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)\sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{-\frac{n^2\overline{\rho}^2}{34n} - \frac{2n\overline{x\rho}}{34n} - \frac{\overline{x}^2}{34n}} \\
&= C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon) e^{-\frac{2\overline{x\rho}}{34}} \sum_{n=1}^\infty \frac{1}{\sqrt{n}} e^{-\frac{n^2\overline{\rho}^2}{34n} - \frac{\overline{x}^2}{34n}} =: C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon) e^{-\frac{2\overline{x\rho}}{34}} \sum_{n=1}^\infty f(n).
\end{aligned}$$

One can check that the function f(z) has one maximum, say at $z = z^*$, so we find

$$\sum_{n=1}^{\infty} J_1(n) \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)e^{-\frac{2\overline{x}\overline{\rho}}{34}} \left(\int_1^\infty \frac{1}{\sqrt{z}} e^{-\frac{z\overline{\rho}^2}{34}-\frac{\overline{x}^2}{34z}}\, dz + f(z^*)\right).$$

We observe that $f(z^*)e^{-\frac{2\overline{x}\overline{\rho}}{34}} \le 1$ holds and that the integral, which can be calculated explicitly, is bounded above by $C(\rho,k)e^{-\frac{\overline{\rho}^2+\overline{x}^2}{34}}$. Hence, we find

$$\sum_{n=1}^{\infty} J_1(n) \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon). \tag{8}$$

Considering now $J_2(n)$, we get, similarly as for J_1, by an application of Corollary 2.2 and the Cauchy-Schwarz inequality

$$\begin{aligned}
J_2(n) &\le \int_{-4\overline{\rho}}^{4\overline{\rho}} C(\rho,k)\Delta\tilde{\Delta}\frac{e^{-\frac{(z-\overline{x}-n\overline{\rho})^2}{2n}}}{\sqrt{2\pi n}}\left(-\ln\epsilon + \left|\ln\left(\max(|\overline{Y}_n|,|z|)\wedge 1\right)\right|\right) dz \\
&\le C(\rho,k)\Delta\tilde{\Delta}\int_{-4\overline{\rho}}^{4\overline{\rho}} \frac{1}{\sqrt{n}} e^{-\frac{(z-\overline{x}-n\overline{\rho})^2}{2n}} \left(-\ln\epsilon + |\ln(|z|\wedge 1)|\right) dz \\
&\le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)C(\rho)\sqrt{\int_{-4\overline{\rho}}^{4\overline{\rho}} \frac{1}{n} e^{-\frac{(z-\overline{x}-n\overline{\rho})^2}{n}}\, dz} \\
&\le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)\sqrt{8\overline{\rho}\frac{1}{n}e^{8\overline{\rho}^2+2|\overline{x}|\overline{\rho}-\frac{\overline{x}^2+n^2\overline{\rho}^2-8\overline{\rho}|\overline{x}|}{n}}}.
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} J_2(n) \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)e^{|\overline{x}|\overline{\rho}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{\overline{x}^2+n^2\overline{\rho}^2-8\overline{\rho}|\overline{x}|}{2n}}. \tag{9}$$

We now distinguish two cases:
Case 1. $\overline{x}^2 \le 8\overline{\rho}|\overline{x}| \Leftrightarrow |\overline{x}| \le 8\overline{\rho}$.
Here, one obviously has

$$\sum_{n=1}^{\infty} J_2(n) \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon). \tag{10}$$

On the other hand we have
Case 2. $\overline{x} > 8\overline{\rho} \Rightarrow A := |\overline{x}|^2 - 8\overline{\rho}|\overline{x}| > 0$.
We first consider the sum on the r.h.s. of (9) and find

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{A+n^2\overline{\rho}^2}{2n}} &\le e^{-\sqrt{A}\overline{\rho}} + \int_1^\infty e^{-\frac{A+z^2\overline{\rho}^2}{2z}}\, dz \\
&= e^{-\sqrt{A}\overline{\rho}} + \sqrt{\frac{\pi}{2}}\frac{1}{\overline{\rho}}\left(e^{\sqrt{A}\overline{\rho}}\operatorname{erfc}\left(\frac{\overline{\rho}+\sqrt{A}}{\sqrt{2}}\right) + e^{-\sqrt{A}\overline{\rho}}\operatorname{erfc}\left(\frac{\overline{\rho}-\sqrt{A}}{\sqrt{2}}\right)\right) \\
&= \le C(\rho)e^{-\sqrt{A}\overline{\rho}}.
\end{aligned}$$

This yields

$$\sum_{n=1}^{\infty} J_2(n) \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)e^{\overline{\rho}(|\overline{x}|-\sqrt{A})} = C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon)e^{\overline{\rho}|\overline{x}|\left(1-\sqrt{1-\frac{8\overline{\rho}}{\overline{x}}}\right)} \le C(\rho,k)\Delta\tilde{\Delta}(-\ln\epsilon). \tag{11}$$

Finally, (6),(7),(8),(10),(11) and Corollary 2.2 prove our assertion. □

Lemma 2.3, 2.4, using the Euclidean norm instead of the maximum norm in $\mathbf{R}^2$ (which gives just an additional constant), expressing the r.h.s. again by the (x, y) variable as well as translation invariance provide

Corollary 2.3

$$\mathbf{E}\left[\int_0^\infty \mathbf{1}_{[z-\Delta,z+\Delta)\times[w-\Delta,w+\Delta)}(X_s,Y_s)\,ds\right] \le \frac{C(\rho,k)}{\epsilon}\Delta^2\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\wedge 1\right)\right| - \ln\epsilon\right)$$

Combining now Lemma 2.2 with the previous corollary, and the limit $\Delta \to 0$, yields

Corollary 2.4

$$W(x,y) = \mathbf{E}\left[\int_0^\infty f(Z_t)\,dt\right] \le \frac{C(\rho,k)}{\epsilon}\int_{\mathbf{R}^2} f(z,w)\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\wedge 1\right)\right| - \ln\epsilon\right)\,dz\,dw.$$

Remark 2.2 *The two previous corollaries mean that, if a fundamental solution for our operator exists, it has an upper bound with a logarithmic singularity at the pole and bounded at infinity. Let us mention that there exist a lot of results on the existence and estimation of fundamental solutions of elliptic PDE's, see e.g. [14], [8] or [6]. But most of them are concerned with operators in divergence form, or operators with only second order terms. A notable exception are the articles [3], for dimension $n \ge 3$, and [4] for $n = 2$ (see the introduction of [3] for a comprehensive overview on this topic). There also non homogeneous operators are considered. Nevertheless, our operator does not belong to the class considered in [4], since the condition on the divergence of the lower order term (equation (5) there) is not fulfilled in our case.*

So, to the best of our knowledge, there does not exist a result in the literature, which would be applicable in our case.

We now arrive at the final estimate for $W(x,y)$, i.e.

Proposition 2.2

$$W(x,y) \le \frac{C(\rho,k)}{\epsilon}\left(||f||_{L^1(G)} + \sqrt{\epsilon}(-\ln\epsilon)||f||_{L^2(G)}\right).$$

Proof. Defining $H := \{(z,w)|(z-x)^2 + \frac{(w-y)^2}{\epsilon^2} \le 1\}$ and splitting the integral, one has

$$\int_{\mathbf{R}^2} f(z,w)\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\wedge 1\right)\right| - \ln\epsilon\right)\,dz\,dw = \int_H + \int_{H^c}. \tag{12}$$

Now, for the second integral one clearly gets

$$\int_{H^c} f(z,w)\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\wedge 1\right)\right| - \ln\epsilon\right)\,dz\,dw = (-\ln\epsilon)||f||_{L^1(H^c)}, \tag{13}$$

whereas for the first one, Cauchy-Schwarz yields

$$\begin{aligned}\int_H f(z,w)\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\wedge 1\right)\right| - \ln\epsilon\right)\,dz\,dw \le \\ ||f||_{L^2(H)}\left|\left|\left(\left|\ln\left(\sqrt{(x-z)^2+\frac{(y-w)^2}{\epsilon^2}}\right)\right| - \ln\epsilon\right)\right|\right|_{L^2(H)}.\end{aligned} \tag{14}$$

Finally, by a change of integration variables $r := z-x$, $s := (w-y)/\epsilon$, the last L^2-norm is easily seen to be smaller than $C\sqrt{\epsilon}(-\ln\epsilon)$, which concludes by (12)-(14) and Corollary 2.4 our proof. □

Finally, we give the proof of our main result, namely

Proof of Theorem 2.1

Let

$$w(x,y) := \lim_{R_n\to\infty} w^{(R_n)}, \tag{15}$$

see Corollary 2.1. Clearly, we have $w_{/\partial G} = 0$, and we shall now show that w is a $W^{2,2}_{loc}(G)$-solution of

$$\mathcal{L}w + f = 0. \tag{16}$$

Let H be an arbitrary bounded domain, s.t. $\overline{H} \subset G$. Moreover, let n_0 be large enough, s.t. $\overline{H} \subset G^{(R_n)}$ holds, for $n \geq n_0$. By Theorem 9.30 of [7], one has $w^{(R_n)} \in W^{2,2}_{loc}\left(G^{(R_n)}\right)$, and the a-priori estimate of Theorem 9.11 of [7] yields that $||w^{(R_n)}||_{W^{2,2}(H)}$ is bounded for $n \geq n_0$. Hence, Sobolev's imbedding theorem gives that the convergence in (15) is uniform, and $w(x, y)$ is continuous in $\overline{H}$.

Now, let v be the $W^{2,2}_{loc}(H)$-solution of

$$\mathcal{L}v + f = 0, \text{ on } H,$$
$$v = w, \text{ on } \partial H.$$

We conclude

$$|v - w| = \lim_{n\to\infty} |v - w^{(R_n)}| = \lim_{n\to\infty} \sup_{\partial H} |v - w^{(R_n)}| = 0,$$

by the maximum principle. Hence, $w = v$, and (16) is proved, since H was arbitrary.

We now prove

$$\lim_{x\to\infty, y\to\infty} w(x, y) = 0. \tag{17}$$

We shall show this limit relation for $W(x, y) = \mathbf{E}\left[\int_0^\infty f(Z_t)\, dt\right]$, which is sufficient because of Corollary 2.1.

So let $R > 0$ be arbitrary, and B_R denotes a sphere with radius R, centered at the origin. Moreover, assume $\sqrt{x^2 + y^2} > R$, and denote $\tau_R := \inf\{t|\, |Z_t| = R\}$. We have

$$\mathbf{P}(\tau_R < \infty) \to 0,$$

for $x \to \infty, y \to \infty$, because of our assumption on the drift of the process Z_t. We now split the function W into three parts, i.e.

$$\begin{aligned} W(x, y) &= \mathbf{E}\left[\int_0^\infty f(Z_t)\, dt \mathbf{1}_{\{\tau_R = \infty\}}\right] + \mathbf{E}\left[\int_0^{\tau_R} f(Z_t)\, dt \mathbf{1}_{\{\tau_R < \infty\}}\right] + \mathbf{E}\left[\int_{\tau_R}^\infty f(Z_t)\, dt \mathbf{1}_{\{\tau_R < \infty\}}\right] \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For I_1 we find, using the definition $\hat{f} := f\mathbf{1}_{\{B_R^c\}}$ and Proposition (2.2),

$$I_1 \leq \frac{C(\rho, k)}{\epsilon}\left(||\hat{f}||_{L^1(G)} + \sqrt{\epsilon}(-\ln \epsilon)||\hat{f}||_{L^2(G)}\right).$$

The same estimate holds clearly for I_2. Let $dF(s)$ be the hitting distribution of Z_{τ_R} on ∂B_R, hence, $\int_{\partial B_R} dF(s) = \mathbf{P}(\tau_R < \infty)$. This provides for I_3

$$\begin{aligned} I_3 &= \int_{\partial B_R} dF(s)\mathbf{E}_s\left[\int_0^\infty f(Z_t)\, dt\right] \leq \int_{\partial B_R} dF(s)\left(\frac{C(\rho, k)}{\epsilon}\left(||f||_{L^1(G)} + \sqrt{\epsilon}(-\ln \epsilon)||f||_{L^2(G)}\right)\right) \\ &= \mathbf{P}(\tau_R < \infty)\left(\frac{C(\rho, k)}{\epsilon}\left(||f||_{L^1(G)} + \sqrt{\epsilon}(-\ln \epsilon)||f||_{L^2(G)}\right)\right). \end{aligned}$$

Therefore,

$$\limsup_{x\to\infty, y\to\infty} (I_1 + I_2 + I_3) \leq \frac{C(\rho, k)}{\epsilon}\left(||\hat{f}||_{L^1(G)} + \sqrt{\epsilon}(-\ln \epsilon)||\hat{f}||_{L^2(G)}\right).$$

Using now our integrability condition of f, gives, after $R \to \infty$, $\limsup_{x\to\infty, y\to\infty}(I_1 + I_2 + I_3) = 0$. This finally provides $\lim_{x\to\infty, y\to\infty} W(x, y) = 0$, hence $\lim_{x\to\infty, y\to\infty} w(x, y) = 0$.

Hence, so far we have constructed a $W^{2,2}_{loc}$ solution of the system

$$\begin{aligned} \mathcal{L}w + f &= 0, \\ w_{/\partial G} &= 0, \\ \lim_{x\to\infty, y\to\infty} w(x,y) &= 0. \end{aligned}$$

The final task is to prove uniqueness. So let w_1, w_2 be two solutions, and denote $D := w_1 - w_2$. This yields

$$\begin{aligned} \mathcal{L}D &= 0, \\ D_{/\partial G} &= 0, \\ \lim_{x\to\infty, y\to\infty} D(x,y) &= 0. \end{aligned} \tag{18}$$

Let τ_n be the exit time of $G^{(R_n)}$ and τ the exit time of G, s.t. we have $\lim_{n\to\infty} \tau_n = \tau$ a.s..

As in the proof of Lemma 2.1, we get by Ito-Krylov's formula (since D fulfills the *homogeneous* PDE) $\mathbf{E}\left[D(X_{\tau_n\wedge t}, Y_{\tau_n\wedge t}\right] = D(x,y)$. Since D is bounded, because the w_i are so, we find by $n\to\infty$

$$\mathbf{E}\left[D(X_{\tau\wedge t}, Y_{\tau\wedge t}\right] = D(x,y). \tag{19}$$

Now, one easily checks by Ito's Lemma (see also Proposition 3.1 [10]) that $-e^{-\frac{2\rho}{k}X^\tau_t}$ is a local supermartingale, bounded above and below (here enters our assumption on the set G!). Therefore, this process is a true supermartingale, hence $\lim_{t\to\infty} -e^{-\frac{2\rho}{k}X^\tau_t}$ exists a.s., and $\lim_{t\to\infty} X^\tau_t$ exists a.s. as well. Clearly, on the set $\{\tau=\infty\}$ this limit can not be finite, and we get

$$\begin{aligned} \lim_{t\to\infty} X_t &= \infty, \\ \lim_{t\to\infty} Y_t &= \infty, \end{aligned} \tag{20}$$

on $\{\tau=\infty\}$, since the same considerations hold for the process Y_t as well. All together, $D_{/\partial G}=0$, the third equation of (18) and (20) gives, after $t\to\infty$ in (19), $D(x,y)=0$, which finishes the proof for $f\geq 0$. For the general case we just have to use the decomposition $f = f^+ - f^-$, as well as the linearity of the basic PDE. □

3 A one dimensional optimal control problem

The main purpose of this section is to prove Proposition 2.1. In order to do this, we shall solve an one dimensional optimal control problem, which we formulate now. Let

$$\begin{aligned} Y^{(1)}_s &= y^{(1)} + \int_t^s v^{(1)}\,ds + B^{(2)}_{s-t}, \;\; s\geq t, \\ J(t,y^{(1)},v^{(1)}) &= \mathbf{E}_{y^{(1)}}\left[\int_t^1 \mathbf{1}_A(s)\mathbf{1}_{[-\Delta,\Delta)}(Y^{(1)}_s)\,ds\right] \to \max, \end{aligned} \tag{21}$$

where we want to maximize over all progressively measurable processes $v^{(1)}_s$ with values in $[0,M]$, $M>0$. Moreover, A denotes the set $A = \cup_{k\in I}(a_{2k}, a_{2k+1}) \subset [0,1]$, where $I=\{0,1,2,..m\}$ or $I=\mathbf{N}_0$, and where $a_i < a_{i+1}$.

For this target functional J we define the following value function

Definition 3.1 *We set*

$$V(t,y^{(1)}) := \sup_{v^{(1)}} J(t,y^{(1)},v^{(1)}).$$

Obviously, $V(1,y^{(1)})=0$ holds. We shall estimate this value function by a series of lemmas, where the first one provides some properties of the value function V, for fixed t.

Lemma 3.1 *For fixed* $t\in[0,1)$ *we have:*
a) $V(t,y^{(1)})$ *is strictly monotone increasing for* $y^{(1)}\in(-\infty,-\Delta]$.
b) $V(t,y^{(1)})$ *is strictly monotone decreasing for* $y^{(1)}\in[\Delta,\infty)$.
c) $V(t,y^{(1)})$ *is concave for* $y^{(1)}\in(-\Delta,\Delta)$.

Proof. We first note that, for fixed $y^{(1)}$, $V(t,y^{(1)})$ is monotone non increasing in t, since this holds obviously for J and fixed strategy $v^{(1)}$.

Let now $y_1^{(1)}<y_2^{(1)}\le-\Delta$.
Denote by $Y_s^{(1),y^{(1)}}$ our process $Y_s^{(1)}$, starting at time t at the value $y^{(1)}$. Furthermore, let τ be the stopping time $\tau:=\inf\{s>t|Y_s^{(1),y^{(1)}}=y_2^{(1)}\}\wedge 1$. By the dynamic programming principle (DPP), as it is formulated, e.g., in (3.20) of [17], we find

$$\begin{aligned} V(t,y_1^{(1)}) &= \sup_{v^{(1)}}\mathbf{E}\left[\int_t^\tau \mathbf{1}_A(s)\mathbf{1}_{[-\Delta,\Delta)}(Y_s^{(1),y^{(1)}})\,ds+V(\tau,Y_\tau^{(1),y^{(1)}})\right]=\mathbf{E}\left[V(\tau,Y_\tau^{(1),y^{(1)}})\right]\\ &= \mathbf{E}\left[V(\tau,Y_\tau^{(1),y^{(1)}})\mathbf{1}_{\{\tau<1\}}\right]=\mathbf{E}\left[V(\tau,y_2^{(1)})\mathbf{1}_{\{\tau<1\}}\right]\le\mathbf{E}\left[V(t,y_2^{(1)})\mathbf{1}_{\{\tau<1\}}\right]<V(t,y_2^{(1)}) \end{aligned}$$

Here we have used our assumption on the $y_i^{(1)}$ in the second equality, the boundary condition at time $t=1$ at the third one, and finally, in the last but one inequality our monotonicity property in t. This shows a), and the point b) can be proved analogously.

We come to the proof of concavity. Let us first note that we have

$$0\le V(t,y)-V(s,y)\le\lambda\left([s,t]\cap A\right), \tag{22}$$

for $s<t$ and λ denoting the Lebesgue measure. Indeed, no strategy can generate an occupation time larger than the r.h.s. of (22). Let now $y^{(1)}\in(-\Delta,\Delta)$ and $\eta>0$ small enough, s.t. $(y^{(1)}-\eta,y^{(1)}+\eta)\subset(-\Delta,\Delta)$. Moreover, let τ denote the stopping time $\tau=\inf\{s>t|Y_s^{(1),y^{(1)}}\notin(y^{(1)}-\eta,y^{(1)}+\eta)\}\wedge 1$. In the following we use again the DPP and the notation $Y_s^{(1),y^{(1)},0}$ for our process $Y_s^{(1),y^{(1)}}$, employing the strategy $v^{(1)} \equiv 0$, as well as $p := \mathbf{P}(\tau = 1)$.

$$
\begin{aligned}
& V(t, y^{(1)}) \\
= \quad & \sup_{v^{(1)}} \mathbf{E}\left[\int_t^\tau \mathbf{1}_A(s)\mathbf{1}_{[-\Delta,\Delta)}(Y_s^{(1),y^{(1)}})\,ds + V(\tau, Y_\tau^{(1),y^{(1)}})\right] \\
= \quad & \sup_{v^{(1)}} \mathbf{E}\left[\mathbf{1}_{\{\tau=1\}}\int_t^\tau \mathbf{1}_A(s)\mathbf{1}_{[-\Delta,\Delta)}(Y_s^{(1),y^{(1)}})\,ds + \right.\\
& \left.\mathbf{1}_{\{\tau<1\}}\left(\int_t^\tau \mathbf{1}_A(s)\mathbf{1}_{[-\Delta,\Delta)}(Y_s^{(1),y^{(1)}})\,ds + V(\tau, Y_\tau^{(1),y^{(1)}})\right)\right] \\
\geq \quad & \sup_{v^{(1)}}\left(p\frac{V(t,y^{(1)}+\eta)+V(t,y^{(1)}-\eta)}{2} + \right.\\
& \left.\mathbf{E}\left[\lambda\left([t,\tau]\cap A\right)\mathbf{1}_{\{\tau<1\}} + V(\tau, Y_\tau^{(1),y^{(1)}})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)}}=y^{(1)}+\eta\}} + V(\tau, Y_\tau^{(1),y^{(1)}})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)}}=y^{(1)}-\eta\}}\right]\right) \\
\geq \quad & \sup_{v^{(1)}}\left(p\frac{V(t,y^{(1)}+\eta)+V(t,y^{(1)}-\eta)}{2} + \right.\\
& \left.\mathbf{E}\left[V(t, Y_\tau^{(1),y^{(1)}})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)}}=y^{(1)}+\eta\}} + V(t, Y_\tau^{(1),y^{(1)}})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)}}=y^{(1)}-\eta\}}\right]\right) \\
\geq \quad & p\frac{V(t,y^{(1)}+\eta)+V(t,y^{(1)}-\eta)}{2} + \\
& \mathbf{E}\left[V(t, Y_\tau^{(1),y^{(1)},0})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)},0}=y^{(1)}+\eta\}} + V(t, Y_\tau^{(1),y^{(1)},0})\mathbf{1}_{\{Y_\tau^{(1),y^{(1)},0}=y^{(1)}-\eta\}}\right] \\
= \quad & p\frac{V(t,y^{(1)}+\eta)+V(t,y^{(1)}-\eta)}{2} + V(t,y^{(1)}+\eta)\frac{1-p}{2} + V(t,y^{(1)}-\eta)\frac{1-p}{2} \\
= \quad & \frac{V(t,y^{(1)}+\eta)+V(t,y^{(1)}-\eta)}{2}.
\end{aligned}
$$

Here we have used in in the last but one inequality the relation (22), and in the last but one equality the fact that, with $v^{(1)} \equiv 0$, the probability of hitting the upper barrier is the same as for the lower one. Hence, for fixed t, $V(t, y^{(1)})$ is midpoint concave, which implies, since it is measurable, that it is concave. This finishes the proof of our Lemma. □

In order to solve the extremal problem (21), we shall apply a result from [19] and find

Lemma 3.2 *The extremal problem (21) is solved by*

$$v^{(1),*}(t, y^{(1)}) = M\mathbf{1}\{(t, y^{(1)})|y^{(1)} \leq \phi(t)\},$$

for some measurable function $\phi(t)$, with values in $[-\Delta, \Delta]$, i.e. one has $V(t, y^{(1)}) = J(t, y^{(1)}, v^{(1),*})$.

Proof. The proof is simply an application of Theorem 5.2) of [19]. Indeed, one easily checks that his assumptions formulated in §5.3 are fulfilled, and that the optimal strategy constructed in §5.3.3*b*) coincides with our assertion, applying just our Lemma 3.1. □

In our next auxiliary result we shall replace the function ϕ of the previous Lemma by a step function, and we will show that the corresponding strategy approximates the optimal one arbitrary well. We have

Lemma 3.3 *For all $\eta > 0$ exists an $n \in \mathbf{N}$ and a strategy*

$$\hat{v}(t, y^{(1)}) = M\mathbf{1}_{\{(t,y^{(1)})|y^{(1)}\leq\gamma(t)\}},$$

where $\gamma(t) := \mathbf{1}_{(t_{i-1},t_i]}(t)\gamma_i$, $i = 1, 2, ..., n$ with $\gamma_i \neq \gamma_{i+1}$, $\gamma_i \in [-\Delta, \Delta]$ and $0 = t_0 < t_1 < t_2 < ... < t_n = 1$, s.t.

$$\left|J(0, y^{(1)}, v^{(1),*}) - J(0, y^{(1)}, \hat{v})\right| \leq \eta,$$

holds uniformly in $y^{(1)}$.

Proof. Since the step functions are dense in $L^p(0,1)$, $p \geq 1$, we have:
Given a $\nu > 0$, there exists an $n(p)$ and a step function $\gamma(t) \in [-\Delta, \Delta]$, s.t. $||\gamma(t) - \phi(t)||_{L^p(0,1)} < \nu$ holds. This implies for the difference of the strategies $v^{(1),*}$ and $\hat{v}$

$$\begin{aligned} ||v^{(1),*} - \hat{v}||_{L^{q,r}} &:= \left(\int_0^1 \left(\int_{\mathbf{R}} \left| v^{(1),*}(s, y^{(1)}) - \hat{v}(s, y^{(1)}) \right|^q dy^{(1)} \right)^{\frac{r}{q}} ds \right)^{\frac{1}{r}} \\ &= M \left(\int_0^1 |\phi(s) - \gamma(s)|^{\frac{r}{q}} ds \right)^{\frac{1}{r}} = M ||\phi - \gamma||_{L^{r/q}}^{1/q} < \nu^{1/q}, \end{aligned} \tag{23}$$

for an $n(r/q)$, where we have used in the first equality the fact that the strategies $v^{(1),*}, \hat{v}$ are $\{0, M\}$-valued. Set now in Proposition 1 of [12], $\sigma \equiv 1, d = 1, b = v^{(1),*}, b^\epsilon = \hat{v}, f = \mathbf{1}_{(-\Delta,\Delta)}, \mu = 0, p = 2$ and $p' = 3$, as well as in Lemma 1 there $i = 2$ and r, q, s.t. $1/(2q) + 1/r < 1/(p \vee i) = 1/2$, e.g. r=q=4. This yields

$$\begin{aligned} & \left| \mathbf{E} \left[\mathbf{1}_{[-\Delta,\Delta)}(Y_t^{(1),y^{(1)},v^{(1),*}}) - \mathbf{1}_{[-\Delta,\Delta)}(Y_t^{(1),y^{(1)},\hat{v}}) \right] \right| \\ \leq \;& C(2,3) ||v^{(1),*} - \hat{v}||_{Y,2,2} \mathbf{E} \left[|\mathbf{1}_{[-\Delta,\Delta)}(y^{(1)} + B_t^{(2)})|^3 \right]^{1/3} \\ \leq \;& C(2,3) C_3 ||v^{(1),*} - \hat{v}||_{L^{4,4}} \leq C(2,3) C_3 \nu^{1/4}, \end{aligned}$$

where we have used Proposition 1 in the first inequality, Lemma 1 for the second one and finally our inequality (23). Now, C_3 depends by Lemma 1 on p, q, r, i and t. But checking the proof of Lemma 1, we see that this "constant" C_3 is increasing in t, hence, we just take its value for $t = 1$, to get a generic constant independent of t. Finally, the constant $C(2,3)$, stemming from Proposition 1, depends on $p, p', \lambda = 1, \Lambda = 1$ and t. Checking the proof, we see that this $t-$dependence comes from the difference of the stochastic exponential of $\hat{v}$ and $v^{(1),*}$, measured in the L^p-norm. Lemma 11 there shows that this constant is again growing in t, s.t. we just take again its value for $t = 1$.

This implies - taking ν small enough -

$$\left| \mathbf{E} \left[\mathbf{1}_{[-\Delta,\Delta)}(Y_t^{(1),y^{(1)},v^{(1),*}}) - \mathbf{1}_{[-\Delta,\Delta)}(Y_t^{(1),y^{(1)},\hat{v}}) \right] \right| \leq \eta,$$

uniformly in $t, y^{(1)}$, hence

$$\left| J(0, y^{(1)}, v^{(1),*}) - J(0, y^{(1)}, \hat{v}) \right| \leq \eta,$$

as claimed. □

We introduce now the shortcut $\hat{Y}_t := Y_t^{(1),y^{(1)},\hat{v}}$ and the new process

$$\tilde{Y}_t := \hat{Y}_t - \gamma_{i+1}, \quad t \in (t_i, t_{i+1}], \; i = 0, 1, 2, ..., n-1,$$

as well as $\tilde{Y}_0 := \hat{Y}_0 = y^{(1)} = y^{(1)} - \gamma_0$. (It will be convenient to set $\gamma_0 = 0$.)

Remark 3.1 *Note that the drift of* $\tilde{Y}_t$ *is given by* $M\mathbf{1}_{(-\infty,0]}(\tilde{Y}_t)$*, but the process jumps at the times* t_i*. The jump heights at the* t_i *are given by* $\gamma_i - \gamma_{i+1}$*, i.e. we have* $\tilde{Y}_{t_i+} - \tilde{Y}_{t_i} = \gamma_i - \gamma_{i+1}$*,* $i = 0, 1, 2, ..., n-1$*.*

One obviously has

Lemma 3.4

$$\sup_{t \in [0,1]} \left| \hat{Y}_t - \tilde{Y}_t \right| \leq \Delta,$$

uniformly in $y^{(1)}$*.*

Finally we define the process

$$\check{Y}_t := y^{(1)} + \int_0^t M\mathbf{1}_{(-\infty,0]}(\check{Y}_s)\, ds + B_t^{(2)},$$

which has the same drift as $\tilde{Y}_t$, but has no jumps. Our next lemma concerns the difference between these two processes, i.e. we claim for $D_t := \tilde{Y}_t - \check{Y}_t$

Lemma 3.5

$$\sup_{t\in[0,1]} |D_t| \leq 3\Delta,$$

uniformly in $y^{(1)}$.

Proof. We first note that one has $D_{t_0-} = 0$ and $D_{t_0+} = -\gamma_1$. We have
Fact 1: If $-\gamma_1 < 0$, $\tilde{Y}_{t_0+} < \check{Y}_{t_0+} = \check{Y}_{t_0}$, the drift of $\tilde{Y}_t$ is at least as large as that of $\check{Y}_t$, hence, in the interval $(t_0, t_1]$, $|\tilde{Y}_t - \bar{Y}_t|$ is decreasing, but $(\tilde{Y}_t - \check{Y}_t)$ keeps its sign. The same is true for $-\gamma_1 > 0$. If D_t reaches the value zero in the interval (t_0, t_1), it remains there until the next jump.

As this holds as well for the consecutive intervals, Fact 1 implies

$$\sup_{t\in[0,1]} |D_t| \leq \max_i \max\left\{|D_{t_i}|, |D_{t_i+}|\right\}. \tag{24}$$

Denote now

$$\begin{aligned}
s_0 &:= D_{t_0-} = 0,\\
s_1 &:= D_{t_0+} = \gamma_0 - \gamma_1 = -\gamma_1,\\
s_2 &:= D_{t_1-} \in [0, -\gamma_1], \ \text{ or } \in [-\gamma_1, 0] \text{ ,depending on the sign of } \gamma_1,\\
s_3 &:= D_{t_1+} = D_{t_1-} + (\gamma_1 - \gamma_2),\\
&...
\end{aligned}$$

and define the d_i by $s_0 = d_0, s_1 = s_0 + d_1, s_2 = s_1 + d_2, s_3 = s_2 + d_3$,..., hence, the $d's$ with odd index are the jumps $\gamma_r - \gamma_{r+1}$. Clearly, by Fact 1, the s_k can change the sign only by d_k's with odd index.

We now claim

$$\sup_k |s_k| \leq 3\Delta, \tag{25}$$

and prove only $\sup_k s_k \leq 3\Delta$, as the proof of the remaining inequality works analogously.

We argue by contradiction. So assume that there exists a k_0, s.t. $s_{k_0} > 3\Delta$. Let now $\tau := \sup\{l < k_0 | s_l \leq 0\}$. By Fact 1 and the definition of the d_i's, τ has to be even, i.e. $\tau = 2m$. This implies

$$d_{\tau+1} + d_{\tau+2} + ... + d_{k_0} \geq 3\Delta.$$

As the $d's$ with even index on the l.h.s. are non positive, this gives

$$d_{\tau+1} + d_{\tau+3} + ... + d_\lambda \geq 3\Delta,$$

where λ is the largest odd index less or equal to k_0. Hence,

$$(\gamma_r - \gamma_{r+1}) + (\gamma_{r+1} - \gamma_{r+2}) + ... + (\gamma_s - \gamma_{s+1}) \geq 3\Delta,$$

for some natural numbers r, s, clearly a contradiction, proving (25) and our Lemma. □

Lemma 3.4 and Lemma 3.5 provide now

Lemma 3.6

$$\sup_{t\in[0,1]} \left|\hat{Y}_t - \check{Y}_t\right| \leq 4\Delta,$$

uniformly in $y^{(1)}$.

The process $\check{Y}_t$ is a Brownian motion with two-valued drift, the transition density $p_t(x, y)$ of which is known in terms of multiple integrals. We exploit this knowledge to give an estimate of $p_t(0, 0)$. i.e.

Lemma 3.7 *We have*

$$p_t(0,0) \le \frac{2}{\sqrt{t}}, \qquad t \in [0,1].$$

Proof. By [11], 6.5.12 and Problem 3.5.8, one has

$$p_t(0,0) = 2\int_0^\infty \int_0^t e^{2b\theta_1} \frac{|b|}{\sqrt{2\pi(t-\tau)^3}} e^{-\frac{(b+M(t-\tau))^2}{2(t-\tau)}} \frac{|b|}{\sqrt{2\pi\tau^3}} e^{-\frac{b^2}{2\tau}}\, db\, d\tau.$$

Introducing the new integration variables $s = \tau/t, a = b - Mt(1-s)s$, one gets - after some calculations -

$$p_t(0,0) = \frac{1}{\pi t^2}\int_0^1 ds \int_{-Mt(1-s)s}^\infty \frac{(a+Mt(1-s)s)^2}{\sqrt{s^3(1-s)^3}} e^{-\frac{a^2}{2t(1-s)s} - \frac{M^2 t(1-s)^2}{2}}\, da.$$

The inner integral can be calculated explicitly, and the result is

$$p_t(0,0) = \int_0^1 (J_{11} + J_{12} + J_{21} + J_{22} + J_3)\, ds. \tag{26}$$

We now estimate the integrals and start with J_1 and J_2. We have

$$|J_{11}| = \frac{\sqrt{t}M^2}{\sqrt{2\pi}}(s-s^2)e^{-\frac{M^2 t(1-s)^2}{2}} erf\left(\frac{M\sqrt{ts}\sqrt{1-s}}{\sqrt{2}}\right),$$

$$|J_{12}| = \frac{\sqrt{t}M^2}{\sqrt{2\pi}}(s-s^2)e^{-\frac{M^2 t(1-s)^2}{2}},$$

hence $|J_{11}| + |J_{12}| \le 2J_{12}$. We now introduce now $g(\alpha) := \int_0^1 (s-s^2)e^{-\alpha(1-s)^2}\, ds$, and get

$$\int_0^1 |J_{12}(s)|\, ds = \sqrt{\frac{2}{\pi t}} g(\frac{M^2 t}{2})\frac{M^2 t}{2}.$$

One easily checks that $\alpha g(\alpha) \le 1/2$, for all non negative α. All together, we arrive at

$$\int_0^1 |J_{11}(s)| + |J_{12}(s)|\, ds \le \sqrt{\frac{2}{\pi t}}. \tag{27}$$

Similarly, one has

$$\int_0^1 |J_{22}(s)|\, ds = \sqrt{\frac{1}{2\pi t}} \int_0^1 e^{-\frac{M^2 t(1-s)^2}{2}}\, ds \le \sqrt{\frac{1}{2\pi t}}.$$

As $|J_{21}(s)| = |J_{22}(s)|\ |erf(\cdot)|$, one arrives at

$$\int_0^1 |J_{21}(s)| + |J_{22}(s)|\, ds \le \sqrt{\frac{2}{\pi t}}. \tag{28}$$

We finally find for J_3

$$\begin{aligned}\int_0^1 |J_3(s)|\, ds &= \frac{M}{\pi}\int_0^1 \sqrt{s(1-s)}e^{-\frac{M^2 t}{2}(1-s)}\, ds \le \frac{M}{\pi}\int_0^1 \sqrt{1-s}e^{-\frac{M^2 t}{2}(1-s)}\, ds \\ &=: \frac{1}{\pi}\sqrt{\frac{2}{t}} M\sqrt{\frac{t}{2}} h(M\sqrt{\frac{t}{2}}) \le \frac{1}{\sqrt{2t}\pi}.\end{aligned} \tag{29}$$

Here we have used the fact that one has for the function h, defined in the last equality, $\alpha h(\alpha) \le 1/2$, for positive α, which one can easily check. Combining (26)-(29), concludes the proof. □

We now use the previous Lemma to estimate the occupation time for the square $[-\Delta, \Delta]^2$ of the two dimensional process $(\check{X}_t, \check{Y}_t)$, where we identify $\check{X}_t \equiv \overline{X}_t$, with $\overline{X}_t$ defined in (5).

Proposition 3.1 *For Δ small enough, say $\Delta < \Delta_0(M, x^{(1)}, y^{(1)})$, we have*

$$\mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\, ds\right] \le C(\overline{\rho})\Delta^2 \left(\ln M - \ln\left(\max(|x^{(1)}|, |y^{(1)}|)\right)\right).$$

Proof. Denoting the transition densities of $\check{X}$ and $\check{Y}$ by $p^{\check{X}}$, resp. $p^{\check{Y}}$, one finds

$$\begin{aligned}
& \lim_{\Delta\to 0}\frac{1}{4\Delta^2}\mathbf{E}\left[\int_0^1 ds\mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\right] \\
= & \lim_{\Delta\to 0}\frac{1}{4\Delta^2}\int_0^1 ds\mathbf{P}(-\Delta\le\check{X}_s<\Delta)\mathbf{P}(-\Delta\le\check{Y}_s<\Delta) \\
= & \lim_{\Delta\to 0}\frac{1}{4\Delta^2}\int_0^1 ds\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta}dz_1dz_2\, p_s^{\check{X}}(x^{(1)},z_1)p_s^{\check{Y}}(y^{(1)},z_2) \\
= & \int_0^1 ds\lim_{\Delta\to 0}\frac{1}{4\Delta^2}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta}dz_1dz_2\, p_s^{\check{X}}(x^{(1)},z_1)p_s^{\check{Y}}(y^{(1)},z_2) \\
= & \int_0^1 ds\, p_s^{\check{X}}(x^{(1)},0)p_s^{\check{Y}}(y^{(1)},0) \\
=: & \kappa^{(1)}(x^{(1)},y^{(1)}) < \kappa^{\infty}(x^{(1)},y^{(1)}) := \int_0^1 ds\, p_s^{\check{X}}(x^{(1)},0)p_s^{\check{Y}}(y^{(1)},0).
\end{aligned}$$

Note that the interchange of the limit and integration is allowed, since we have by the seminal paper of [1], Theorem 7 (and the remark on page 609 that his condition H is satisfied for uniformly parabolic operators with bounded coefficients), estimates for the the fundamental solutions, which are nothing else but the product of the transition densities, which provide an integrable majorant.

Employing now Lemma 4.1, we find

$$\mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\,ds\right]\le C(\overline{\rho})\Delta^2\left(\ln M-\ln\left(\max(|x^{(1)}|,|y^{(1)}|)\right)\right),$$

for Δ small enough, say $\Delta<\Delta_0(M,x^{(1)},y^{(1)})$, and $|x^{(1)}|,|y^{(1)}|\le 1$.

For $(x^{(1)},y^{(1)})\notin Q_1:=[-1,1]^2$, let $\tau:=\inf\{t|(\check{X},\check{Y})_t\in Q_1\}$. We find

$$\begin{aligned}
& \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\,ds\right]=\mathbf{E}\left[\mathbf{1}_{\{\tau<1\}}\int_\tau^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\,ds\right] \\
= & \int_{\partial Q_1} dF(z)\mathbf{E}\left[\mathbf{1}_{\{\tau<1\}}\int_\tau^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\,ds\right] \\
\le & \sup_{z\in\partial Q_1}\mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\Delta,\Delta)}(\check{Y}_s)\,ds\,\big|(\check{X},\check{Y})_0=z\right],
\end{aligned}$$

where $dF(z)$ is the (incomplete) distribution of $(\check{X},\check{Y})_\tau$ on ∂Q_1. This completes the proof, using the result from the case $|x^{(1)}|,|y^{(1)}|\le 1$. □

Finally, we prove the main auxiliary result of section 2, namely, we give the proof of Proposition 2.1 Note that we replace the "local variable" Δ, which we have used in section 3, by $\tilde{\Delta}$ of section 2. Moreover, we use the identity $X_t^{(1)}\equiv\check{X}_t$.

Proof of Proposition 2.1. We first note that

$$\mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_s^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{(1)})\,ds\right]\le\mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{*,A(\check{X})})\,ds\right],$$

where $Y^{*,A(\check{X})}$ stands for solution process of the extremal problem in Lemma 3.2, and where we set $A(\check{X}):=\{t|\check{X}_t\in[-\Delta,\Delta)\}$. We estimate further

$$\begin{aligned}
& \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_s^{*,A(\check{X})})\,ds\right] \\
\le & \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(\hat{Y}_s^{A(\check{X})})\,ds\right]+\eta \\
\le & \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(\check{X}_s)\mathbf{1}_{[-5\tilde{\Delta},5\tilde{\Delta})}(\check{Y}_s)\,ds\right]+\eta \\
\le & C(\overline{\rho})\Delta\tilde{\Delta}\left(-\ln\left(\max(|x^{(1)}|,|y^{(1)}|)\wedge 1\right)+\ln M\right).
\end{aligned}$$

Observe that the first inequality follows by Lemma 3.3, and that η can be chosen independently of the set A, which follows easily by the last paragraph of its proof. Moreover, the second inequality uses Lemma 3.6 and the last one Proposition 3.1, which concludes the proof of a).

For the point b), we define $\tau := \inf\{t||X_t^{(1)}| = |x^{(1)}|/3\}$ and get by the strong Markov property

$$\begin{aligned}
& \mathbf{E}\left[\int_0^1 \mathbf{1}_{[-\Delta,\Delta)}(X_t^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_t^{(1)})\,dt\right] \\
= \; & \mathbf{E}\left[\int_\tau^1 \mathbf{1}_{[-\Delta,\Delta)}(X_t^{(1)})\mathbf{1}_{[-\tilde{\Delta},\tilde{\Delta})}(Y_t^{(1)})\,dt\,\middle|\,|X_\tau^{(1)}| = |x^{(1)}|/3\right]\mathbf{P}(\tau<1) \\
& \le C(\overline{\rho})\Delta\tilde{\Delta}(\ln M)\mathbf{P}(\tau<1).
\end{aligned}$$

Here we have used in the last inequality the result of point a). Moreover, our assumption $|x^{(1)}| > 4\rho_1$, as well as the known distribution of the running maximum of a Brownian motion, see e.g. [11], (2.8.4), yield the crude upper estimate $\mathrm{erfc}(|x^{(1)}|/(4\sqrt{2})) \le \exp(-(x^{(1)})^2/32)$ for $\mathbf{P}(\tau<1)$, which concludes the proof of b).

The proof of c) and d) works analogously to b). □

4 Appendix

Lemma 4.1 *We have*

$$\kappa^\infty(x^{(1)},y^{(1)}) := \int_0^1 ds\, p_s^{\check{X}}(x^{(1)},0)p_s^{\check{Y}}(y^{(1)},0) \le C(\overline{\rho})\left(\ln M - \ln\left(\max(|x^{(1)}|,|y^{(1)}|)\right)\right),$$

for $|x^{(1)}|,|y^{(1)}| \le 1$.

Proof. Let us first note that it suffices to give the prove for the case $y^{(1)} \le 0$, since, due to the structure of the drift of $\check{Y}$, the estimate will hold for the case $y^{(1)} > 0$ a fortiori. So we restrict to $y^{(1)} \le 0$.

Let $dF(r)$ be the distribution of the hitting time of a standard Brownian motion with drift M to the level $|y^{(1)}|$. We have for the one dimensional transition density

$$p_s^{\check{Y}}(y^{(1)},0) = \int_0^s p_{s-r}^{\check{Y}}(0,0)dF(r) \le 2\int_0^s \frac{1}{\sqrt{s-r}}\frac{|y^{(1)}|}{\sqrt{2\pi r^3}}e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}\,dr,$$

where we have used Lemma 3.7 for the last inequality and the well known distribution $dF(r)$. Hence, we find for κ^∞ the estimate

$$\begin{aligned}
\kappa^\infty(x^{(1)},y^{(1)}) &\le 2\int_0^\infty \frac{ds}{\sqrt{2\pi s}}e^{-\frac{(x^{(1)}+\overline{\rho}s)^2}{2s}}\int_0^s \frac{1}{\sqrt{s-r}}\frac{|y^{(1)}|}{\sqrt{2\pi r^3}}e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}\,dr \\
&= C|y^{(1)}|\int_0^\infty dr\frac{e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}}{r^{3/2}}\int_r^\infty ds\frac{1}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)}+\overline{\rho}s)^2}{2s}}. \qquad (30)
\end{aligned}$$

The inner integral can be estimated by - remember $|x^{(1)}| \le 1$ -

$$C(\overline{\rho})\int_r^\infty ds\frac{1}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s}$$

We now distinguish two cases.
Case 1: $r \ge 1$.
Here we find

$$C(\overline{\rho})\int_r^\infty ds\frac{1}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s} \le C(\overline{\rho})\int_r^\infty ds\frac{1}{\sqrt{(s-r)}}e^{-\frac{\overline{\rho}^2}{2}s} \le C(\overline{\rho}). \qquad (31)$$

Case 2: $r < 1$.
We split the integration area into two parts, i.e. we define $C(\overline{\rho})\int_r^\infty = C(\overline{\rho})\int_r^{2r} + C(\overline{\rho})\int_{2r}^\infty =: I_1 + I_2$.
For I_1 we find

$$I_1 = C(\overline{\rho})\int_r^{2r}\frac{ds}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s} \le C(\overline{\rho})\int_r^{2r}\frac{ds}{\sqrt{s(s-r)}} = C(\overline{\rho})\int_1^2\frac{dw}{\sqrt{w(w-1)}} \le C(\overline{\rho}). \tag{32}$$

For I_2 we have, using $s \ge 2r$ and the fact that $C(\overline{\rho})$ may vary,

$$\begin{aligned} I_2 &= C(\overline{\rho})\int_{2r}^\infty\frac{ds}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s} \le C(\overline{\rho})\int_{2r}^\infty ds\frac{e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s}}{s}. \\ &= C(\overline{\rho})\int_{2r}^3 ds\frac{e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s}}{s} + C(\overline{\rho})\int_3^\infty ds\frac{e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s}}{s} \le C(\overline{\rho})(1-\ln(r)). \end{aligned} \tag{33}$$

We also find the different estimate for I_2

$$I_2 \le C(\overline{\rho})\int_0^\infty ds\frac{e^{-\frac{(x^{(1)})^2}{s}-\frac{\overline{\rho}^2}{2}s}}{s} = C(\overline{\rho})K_0(\sqrt{2}|x^{(1)}|\overline{\rho}) \le C(\overline{\rho})(1-\ln|x^{(1)}|).$$

Combining these two estimates yields

$$I_2 \le C(\overline{\rho})\left(1-\ln(r\vee|x^{(1)}|)\right),$$

and all together for the inner integral, and for $r > 0$,

$$C(\overline{\rho})\int_r^\infty ds\frac{1}{\sqrt{s(s-r)}}e^{-\frac{(x^{(1)})^2}{2s}-\frac{\overline{\rho}^2}{2}s} \le C(\overline{\rho})\left(1-\ln((r\vee|x^{(1)}|)\wedge 1)\right). \tag{34}$$

This gives for κ^∞ the upper estimate

$$\begin{aligned} \kappa^\infty(x^{(1)},y^{(1)}) &\le C(\overline{\rho})|y^{(1)}|\left(\int_0^1\left(1-\ln(r\vee|x^{(1)}|)\right)\frac{e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}}{r^{3/2}}\,dr + \int_1^\infty\frac{e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}}{r^{3/2}}\,dr\right) \\ &=: K_{11}+K_{12}+K_{13}+K_2 \end{aligned} \tag{35}$$

For K_2 we have

$$K_2 \le C(\overline{\rho})|y^{(1)}|\int_0^\infty\frac{e^{-\frac{(|y^{(1)}|-Mr)^2}{2r}}}{r^{3/2}}\,dr \le C(\overline{\rho}), \tag{36}$$

as the last integral is equal to $\sqrt{2\pi}/|y^{(1)}|$. Analogously, one finds

$$K_{11} \le C(\overline{\rho}). \tag{37}$$

The remaining terms are

$$\begin{aligned} K_{12} &= C(\overline{\rho})|y^{(1)}|e^{|y^{(1)}|M}(-\ln(|x^{(1)}|))\int_0^{|x^{(1)}|}\frac{e^{-\frac{(y^{(1)})^2}{2r}-\frac{M^2}{2}r}}{r^{3/2}}\,dr \\ K_{13} &= C(\overline{\rho})|y^{(1)}|e^{|y^{(1)}|M}\int_{|x^{(1)}|}^1(-\ln r)\frac{e^{-\frac{(y^{(1)})^2}{2r}-\frac{M^2}{2}r}}{r^{3/2}}\,dr. \end{aligned} \tag{38}$$

We start with the estimate for K_{12} and distinguish two cases:
Case 1: $|x^{(1)}| \ge \frac{|y^{(1)}|^3}{2M}$.
Here we have

$$K_{12} \le C(\overline{\rho})|y^{(1)}|e^{|y^{(1)}|M}(-\ln(|x^{(1)}|))\int_0^\infty\frac{e^{-\frac{(y^{(1)})^2}{2r}-\frac{M^2}{2}r}}{r^{3/2}}\,dr \le C(\overline{\rho})\left(\ln M-\ln(\max(|x^{(1)}|,|y^{(1)}|))\right), \tag{39}$$

where we have used that $|y^{(1)}|\, e^{|y^{(1)}|M} \int_0^\infty \frac{e^{-\frac{(y^{(1)})^2}{2r} - \frac{M^2}{2} r}}{r^{3/2}}\, dr = \sqrt{2\pi}$, and the fact that in Case 1 one checks easily $-\ln|x^{(1)}| \le C\left(\ln M - \ln(\max(|x^{(1)}|, |y^{(1)}|))\right)$.

Case 2: $|x^{(1)}| < \frac{|y^{(1)}|^3}{2M}$.

We start with the observation that one can calculate K_{12} explicitly:

$$\begin{aligned} K_{12} &= C(\overline{\rho})(-ln|x^{(1)}|)\left(\text{erfc}\left(\frac{M|x^{(1)}| + |y^{(1)}|}{\sqrt{|2x^{(1)}|}}\right) e^{2|y^{(1)}|M} + \text{erfc}\left(\frac{-M|x^{(1)}| + |y^{(1)}|}{\sqrt{|2x^{(1)}|}}\right)\right) \\ &=: K_{121} + K_{122}. \end{aligned} \tag{40}$$

For K_{121}, we get

$$\begin{aligned} K_{121} &\le C(\overline{\rho})(-\ln|x^{(1)}|)\exp\left(-\frac{M^2|x^{(1)}|}{2} - \frac{(y^{(1)})^2|}{2|x^{(1)}|}\right)e^{|y^{(1)}|M}\frac{\sqrt{|x^{(1)}|}}{|y^{(1)}|} \\ &\le C(\overline{\rho})(-\ln|x^{(1)}|)\frac{\sqrt{|x^{(1)}|}}{|y^{(1)}|} = C(\overline{\rho})(-\ln|x^{(1)}|)|x^{(1)}|^{1/6}\frac{|x^{(1)}|^{1/3}}{|y^{(1)}|} \le C(\overline{\rho}). \end{aligned} \tag{41}$$

Here we have used the inequality $\text{erfc}(z) \le e^{-z^2}/z$ for positive z in the first inequality, the inequality $a^2 + b^2 \ge 2ab$ for positive a, b in the second one, and finally our assumption in Case 2.

For K_{122}, we have

$$\begin{aligned} K_{122} &\le C(\overline{\rho})(-\ln|x^{(1)}|)\text{erfc}\left(\frac{|y^{(1)}|}{2\sqrt{2|x^{(1)}|}}\right) \le C(\overline{\rho})(-\ln|x^{(1)}|)\exp\left(-\frac{(y^{(1)})^2}{8|x^{(1)}|}\right) \\ &\le C(\overline{\rho})(-\ln|x^{(1)}|)\exp\left(-\frac{1}{8|x^{(1)}|^{1/3}}\right) \le C(\overline{\rho}). \end{aligned} \tag{42}$$

We have used $|y^{(1)}| - |x^{(1)}|M \ge |y^{(1)}|/2$, coming from our assumption in case 2, in the first inequality, $\text{erfc}(z) \le e^{-z^2}$ in the second one, and again our assumption in case 2 for the third one.

Combining (41) and (42), provides

$$K_{12} \le C(\overline{\rho}). \tag{43}$$

Concerning K_{13}, we again distinguish between two cases.

Case 1: $|x^{(1)}| \ge |y^{(1)}|^3/2$.

We find

$$K_{13} \le C(\overline{\rho})|y^{(1)}|\, e^{|y^{(1)}|M}(-\ln|x^{(1)}|)\int_{|x^{(1)}|}^1 \frac{e^{-\frac{(y^{(1)})^2}{2r} - \frac{M^2}{2}r}}{r^{3/2}}\, dr \le C(\overline{\rho})\left(-\ln(\max(|x^{(1)}|, |y^{(1)}|))\right), \tag{44}$$

where we replaced the integral by an integral over $(0, \infty)$, which can be calculated explicitly, for the second inequality. Moreover, we estimated $(-\ln|x^{(1)}|)$ by $C\left(-\ln(\max(|x^{(1)}|, |y^{(1)}|))\right)$, which is true in Case 1.

Case 2: $|x^{(1)}| < |y^{(1)}|^3/2 \Leftrightarrow |(y^{(1)})^2/(2|x^{(1)}|) > 1/|y^{(1)}|$.

Here one gets

$$\begin{aligned}
K_{13} &= C(\overline{\rho})e^{|y^{(1)}|M}\int_{(y^{(1)})^2/2}^{(y^{(1)})^2/(2|x^{(1)}|)} \frac{e^{-w-\frac{(y^{(1)})^2M^2}{4w}}}{w^{1/2}}\Big(-\ln(y^{(1)})^2+\ln(2w)\Big)\,dw \\
&\leq C(\overline{\rho})e^{|y^{(1)}|M}\int_0^\infty \frac{e^{-w-\frac{(y^{(1)})^2M^2}{4w}}}{w^{1/2}}\Big(-\ln(y^{(1)})^2)\Big)\,dw \\
&+ C(\overline{\rho})e^{|y^{(1)}|M}\int_{1/2}^\infty \frac{e^{-w-\frac{(y^{(1)})^2M^2}{4w}}}{w^{1/2}}\ln(2w)\,dw \\
&\leq C(\overline{\rho})\Big(-\ln(\max(|x^{(1)}|,|y^{(1)}|))\Big)+C(\overline{\rho})e^{|y^{(1)}|M}\int_{1/2}^{M/|y^{(1)}|} \frac{e^{-w-\frac{(y^{(1)})^2M^2}{4w}}}{w^{1/2}}\ln(2w)\,dw \\
&+ C(\overline{\rho})e^{|y^{(1)}|M}\int_{M/|y^{(1)}|}^\infty \frac{e^{-w-\frac{(y^{(1)})^2M^2}{4w}}}{w^{1/2}}\ln(2w)\,dw \\
&=: C(\overline{\rho})\Big(-\ln(\max(|x^{(1)}|,|y^{(1)}|))\Big)+L_1+L_2. \qquad (45)
\end{aligned}$$

We have used in the second inequality the fact that the integral over $(0,\infty)$ is explicitly known and our assumption in Case 2. For L_1, one easily gets

$$L_1 \leq C(\overline{\rho})\Big(\ln M - \ln|y^{(1)}|\Big). \qquad (46)$$

For L_2, we have

$$\begin{aligned}
L_2 &\leq C(\overline{\rho})e^{|y^{(1)}|M}\int_{M/|y^{(1)}|}^\infty \frac{e^{-w}}{w^{1/2}}\ln(2w)\,dw \\
&= C(\overline{\rho})e^{|y^{(1)}|M}\int_0^\infty \frac{e^{-z-M/|y^{(1)}|}}{(z+M/|y^{(1)}|)^{1/2}}\ln\left(2z+\frac{2M}{|y^{(1)}|}\right)dz \\
&\leq C\int_0^\infty \frac{e^{-z}}{(z)^{1/2}}\ln\left(2z+\frac{2M}{|y^{(1)}|}\right)dz \leq C(\overline{\rho})\Big(\ln M-\ln(\max(|x^{(1)}|,|y^{(1)}|))\Big), \qquad (47)
\end{aligned}$$

where we have splitted the integral into $\int_0^1$ and $\int_1^\infty$ and used the fact that we are in Case 2 for the last equality. (45)-(47) provide

$$K_{13} \leq C(\overline{\rho})\Big(\ln M - \ln(\max(|x^{(1)}|,|y^{(1)}|))\Big). \qquad (48)$$

(35),(36),(37),(39),(43) and (48) prove our Lemma. □

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