Information-Based Asset Pricing

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(Mathematical Methods for Finance: Educational Workshop)

1. Information-based pricing framework

Information-driven asset-price dynamics

In derivative pricing, the starting point is usually the specification of a model for the price process of the underlying asset.

For example, in the Black-Scholes-Merton theory, the underlying asset has a geometric Brownian motion as its price process.

More generally, the economy is often modelled by a probability space equipped with the filtration generated by a multi-dimensional Brownian motion, and it is assumed that asset prices are adapted to this filtration.

The basic problem with this approach is that the market filtration is fixed, and no comment is offered on the issue of “where it comes from”.

In other words, the filtration, which represents the revelation of information to market participants, is modelled first, in an *ad hoc* manner, and then it is assumed that the asset price processes are adapted to it.

But no indication is given about the nature of this “information”, and it is not obvious why the Brownian filtration is providing information rather than noise.

In a complete market there is a sense in which the Brownian filtration provides no irrelevant information.

Nevertheless, the notion that the market filtration should be "prespecified" is an unsatisfactory one in financial modelling.

What is unsatisfactory about the "prespecified-filtration" is that little structure is given to the filtration: price movements behave as though they were spontaneous.

In reality, we expect the price-formation process to exhibit more structure.

It would be out of place to attempt an account of the process of price formation—nevertheless, we can improve on the "prespecified" approach.

In that spirit we proceed as follows.

We note that price changes arise from two sources.

The first is that resulting from changes in agent preferences—that is to say, changes in the pricing kernel.

Movements in the pricing kernel are associated with (a) changes in investor attitudes towards risk, and (b) changes in investor "impatience", the subjective discounting of future cash flows.

Equally important are changes in price resulting from the revelation of information about the future cash flows derivable from a given asset.

When a market agent decides to buy or sell an asset, the decision is made in accordance with the information available to the agent concerning the likely future cash flows associated with the asset.

A change in the information available to the agent about a future cash flow will typically have an effect on the price at which they are willing to buy or sell, even if the agent's preferences remain unchanged.

The movement of the price of an asset should, therefore, be regarded as *an emergent phenomenon*.

To put the matter another way, the price process of an asset should be viewed as the output of (rather than an input into) the decisions made relating to possible transactions in the asset, and these decisions should be understood as being induced primarily by the flow of information to market participants.

Taking into account this observation we propose a new framework for asset pricing based on *modelling of the flow of market information*.

Mathematical framework

The framework discussed here will be based on modelling the flow of partial information to market participants about impending debt obligation and equity dividend payments.

As usual, we model the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$.

The probability measure $\mathbb{Q}$ is understood to be the risk-neutral measure, and the filtration $\{\mathcal{F}_t\}$ is understood to be the "market filtration".

Thus all asset-price processes and other information-providing processes accessible to market participants are assumed to be adapted to $\{\mathcal{F}_t\}$.

We assume the absence of arbitrage and the existence of a pricing kernel.

With these conditions the existence of a unique risk-neutral measure is ensured.

We assume that the default-free discount-bond system, denoted by $\{P_{tT}\}_{0\leq t\leq T<\infty}$, can be written in the form

$$P_{tT} = \frac{P_{0T}}{P_{0t}}. \tag{1}$$

It follows that if the random variable D_T represents a cash flow occurring at time T, then its value S_t at any earlier time t is given by

$$S_t = P_{tT}\mathbb{E}\left[D_T|\mathcal{F}_t\right]. \tag{2}$$

This is the discounted conditional expectation of D_T in the risk-neutral measure.

In the case where the asset pays a sequence of dividends D_{T_k} $(k = 1, 2, \ldots, n)$ on the dates T_k the price (for $t < T_1$) is given by

$$S_t = \sum_{k=1}^{n} P_{tT_k}\mathbb{E}\left[D_{T_k}|\mathcal{F}_t\right]. \tag{3}$$

More generally, for all $t \geq 0$, and taking into account the ex-dividend behaviour, we have

$$S_t = \sum_{k=1}^{n} \mathbb{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}\left[D_{T_k} | \mathcal{F}_t\right] . \qquad (4)$$

Modelling the flow of information

For the moment we consider the case in which the asset entails a single payment D_T at time T.

We make the reasonable assumption that *some* partial information regarding the value of the cash flow D_T is available at earlier times.

This information will in general be imperfect.

The model for such imperfect information will be of a simple type that allows for a great deal of analytic tractability.

In this model, information about the true value of the cash flow steadily increases, while at the same time the obscuring factors at first increase in magnitude, and then eventually die away just as the payment day.

More precisely, we shall assume that the following $\{\mathcal{F}_t\}$-adapted market information process is accessible to market participants:

$$\xi_t = \sigma t D_T + \beta_{tT}. \tag{5}$$

Here the process $\{\beta_{tT}\}_{0\leq t\leq T}$ is a standard Brownian bridge on the interval $[0, T]$.

We assume that $\{\beta_{tT}\}$ is independent of D_T, and thus represents "pure noise".

The Brownian bridge process satisfies $\beta_{0T} = 0$, and $\beta_{TT} = 0$ (see Figure 1).

We also have $\mathbb{E}[\beta_{tT}] = 0$ and

$$\mathbb{E}\left[\beta_{sT}\beta_{tT}\right] = \frac{s(T-t)}{T} \tag{6}$$

for all s, t satisfying $0 \leq s \leq t \leq T$.

The market participants do not have direct access to the bridge process $\{\beta_{tT}\}$.

That is to say, $\{\beta_{tT}\}$ is not assumed to be adapted to $\{\mathcal{F}_t\}$.

We can thus think of $\{\beta_{tT}\}$ as representing the rumour, speculation, misrepresentation, overreaction, and general disinformation often occurring, in one form or another, in connection with financial activity.

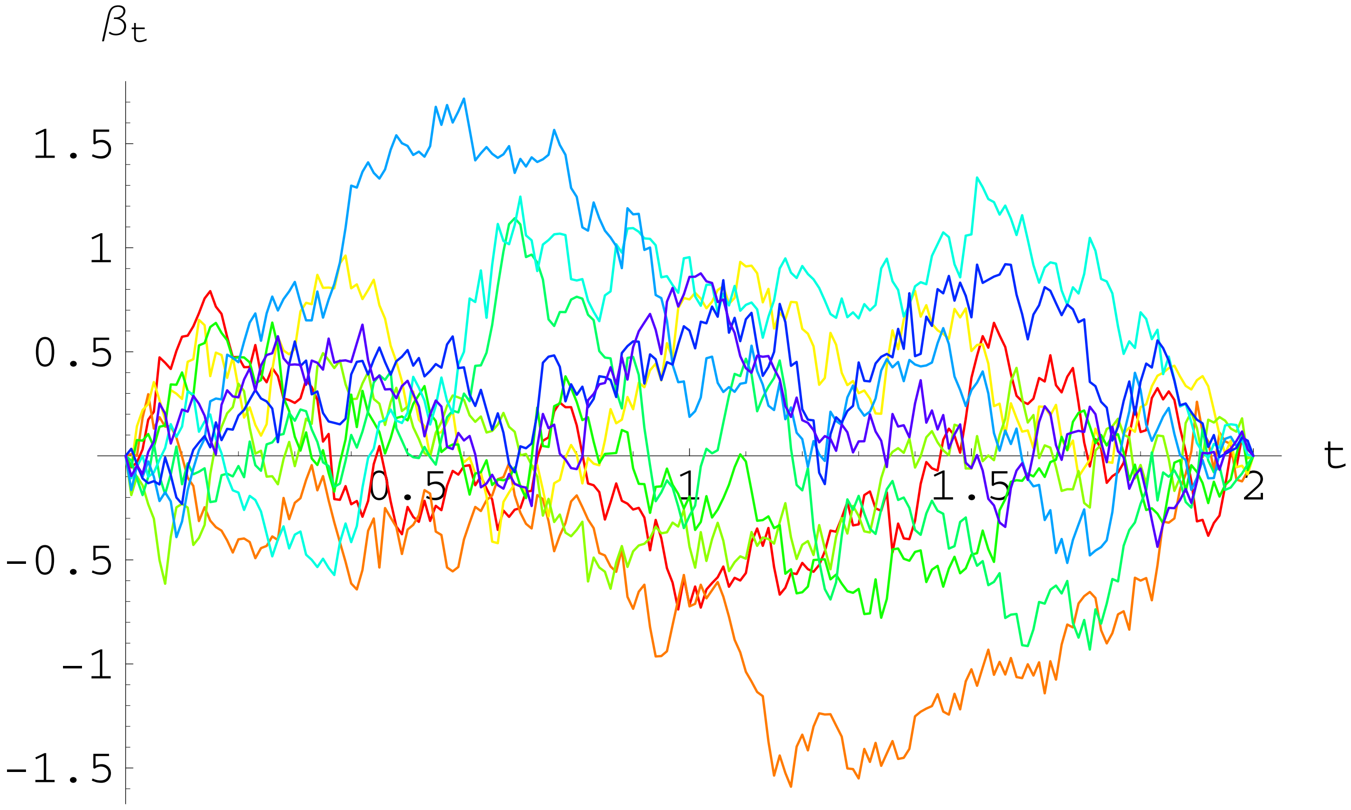

Figure 1: Sample paths for the Brownian bridge over the time period $[0, 2]$.

The parameter σ represents the rate at which the true value of D_T is "revealed" as time progresses.

If σ is low, then the true value of D_T is effectively hidden until very near the payment date of the asset.

On the other hand, if σ is high, then D_T is revealed quickly.

On Markovian nature of the information process

More generally, the rate at which the true value of D_T is revealed is not constant.

In that case we will have

$$\xi_t = D_T \int_0^t \sigma_s \mathrm{d}s + \beta_{tT}, \tag{7}$$

where $\sigma_s \geq 0$.

When $\{\sigma_t\}$ is constant, the resulting information process (5) is Markovian.

Along with the fact that D_T is $\mathcal{F}_T$-measurable we thus find that

$$\mathbb{E}\left[D_T|\mathcal{F}_t\right] = \mathbb{E}\left[D_T|\xi_t\right], \tag{8}$$

which simplifies the calculation.

For the moment we shall assume that $\sigma_t = \sigma$ is constant.

Determining the conditional expectation

If the random variable D_T that represents the payoff has a continuous distribution, then the conditional expectation in (8) can be expressed as

$$\mathbb{E}\left[D_T|\xi_t\right] = \int_0^\infty x\pi_t(x)\,\mathrm{d}x. \tag{9}$$

Here $\pi_t(x)$ is the conditional probability density for the random variable D_T:

$$\pi_t(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{Q}(D_T \leq x|\xi_t). \tag{10}$$

We assume appropriate technical conditions on the distribution of the dividend that will suffice to ensure the existence of the expressions under consideration.

Also, for convenience we use a notation appropriate for continuous distributions, though corresponding results can be inferred for discrete distributions, or more general distributions, by a slight modification.

Bearing these in mind, the conditional probability density process for the dividend can be worked out by use of a form of the Bayes formula:

$$\pi_t(x) = \frac{p(x)\rho(\xi_t|D_T = x)}{\int_0^\infty p(x)\rho(\xi_t|D_T = x)\mathrm{d}x}. \tag{11}$$

Here $p(x)$ denotes the *a priori* probability density for D_T, which we assume is known as an initial condition, and $\rho(\xi_t|D_T = x)$ denotes the conditional density for the random variable ξ_t given that $D_T = x$.

Since β_{tT} is a Gaussian random variable with mean zero and variance $t(T-t)/T$, we deduce that the conditional probability density for ξ_t is

$$\rho(\xi_t|D_T = x) = \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)}\right). \tag{12}$$

Inserting this expression into the Bayes formula we get

$$\pi_t(x) = \frac{p(x)\exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right]}{\int_0^\infty p(x)\exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right]\mathrm{d}x}. \tag{13}$$

We thus obtain the following result for the asset price:

The information-based price process $\{S_t\}_{0\leq t\leq T}$ of a limited-liability asset that pays a single dividend D_T at time T with a priori distribution

$$\mathbb{Q}(D_T \leq y) = \int_0^y p(x)\,\mathrm{d}x \tag{14}$$

is given by

$$S_t = \mathbb{1}_{\{t<T\}} P_{tT} \frac{\int_0^\infty x p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx}, \tag{15}$$

where $\xi_t = \sigma t D_T + \beta_{tT}$ is the market information.

Asset price dynamics in the case of a single cash flow

In order to analyse the properties of the price process deduced above, and to be able to compare it with other models, we need to work out its dynamics.

One of the advantages of the framework under consideration is that we have an explicit expression for the price at our disposal.

Thus in obtaining the dynamics we need to find the stochastic differential equation of which $\{S_t\}$ is the solution.

Let us write

$$D_{tT} = \mathbb{E}[D_T|\xi_t]. \tag{16}$$

Evidently, D_{tT} can be expressed in the form $D_{tT} = D(\xi_t, t)$, where $D(\xi, t)$ is defined by

$$D(\xi, t) = \frac{\int_0^\infty xp(x) \exp\left[\frac{T}{T-t}(\sigma x\xi - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x\xi - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}. \tag{17}$$

A straightforward calculation making use of the Ito rules shows that the dynamical equation for $\{D_{tT}\}$ is given by

$$\mathrm{d}D_{tT} = \frac{\sigma T}{T-t} V_t \left[\frac{1}{T-t}\Big(\xi_t - \sigma T D_{tT}\Big)\mathrm{d}t + \mathrm{d}\xi_t\right]. \tag{18}$$

Here V_t is the conditional variance of the dividend:

$$V_t = \mathbb{E}_t\left[\left(D_T - \mathbb{E}_t[D_T]\right)^2\right] = \int_0^\infty x^2 \pi_t(x)\,\mathrm{d}x - \left(\int_0^\infty x\pi_t(x)\,\mathrm{d}x\right)^2. \tag{19}$$

Therefore, if we define a new process $\{W_t\}_{0\leq t<T}$ by setting

$$W_t = \xi_t - \int_0^t \frac{1}{T-s}\Big(\sigma T D_{tT} - \xi_s\Big)\mathrm{d}s, \tag{20}$$

we find, after some rearrangement, that

$$\mathrm{d}D_{tT} = \frac{\sigma T}{T-t} V_t \mathrm{d}W_t. \tag{21}$$

For the dynamics of the asset price we thus have

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \Gamma_{tT}\mathrm{d}W_t, \tag{22}$$

where $r_t = -\mathrm{d}\ln P_{0t}/\mathrm{d}t$ is the short rate, and the absolute price volatility Γ_{tT} is

$$\Gamma_{tT} = P_{tT}\frac{\sigma T}{T-t} V_t. \tag{23}$$

As we shall demonstrate later, the process $\{W_t\}$ defined in (20) is an $\{\mathcal{F}_t\}$-Brownian motion.

Hence *from the point of view of the market it is the process* $\{W_t\}$ *that drives the asset price dynamics*.

In this way our framework resolves the paradoxical point of view usually adopted in financial modelling in which $\{W_t\}$ is regarded on the one hand as "noise", and yet on the other hand also generates the market information flow.

Therefore, instead of hypothesising the existence of a driving process for the dynamics of the markets, we are able to *deduce* the existence of such a process.

2. Black-Scholes theory from the information perspective

From cash flow to market factors

In general the cash flow D_T is determined by a number of independent market factors (called the 'X factors'):

$$D_T = D(X_T^1, X_T^2, \ldots, X_T^n). \tag{24}$$

For each market factor we have the vector-valued information process

$$\xi_{tT}^{\alpha} = \sigma^{\alpha} X_T^{\alpha} t + \beta_{tT}^{\alpha}, \tag{25}$$

which generates the market filtration.

Log-normal cash flow

The simplest application of the X-factor technique arises in the case of geometric Brownian motion models.

We consider a limited-liability company that makes a single cash distribution D_T at time T.

We assume that D_T has a log-normal distribution under $\mathbb{Q}$, and thus can be written in the form

$$D_T = S_0 \exp\left(rT + \nu\sqrt{T}X_T - \tfrac{1}{2}\nu^2 T\right), \tag{26}$$

where the market factor X_T is normally distributed with mean zero and variance one, and $r > 0$ and $\nu > 0$ are constants.

The information process $\{\xi_t\}$ is taken to be of the form

$$\xi_t = \sigma t X_T + \beta_{tT}, \tag{27}$$

where the Brownian bridge $\{\beta_{tT}\}$ is independent of X_T, and where the information flow rate is of the special form

$$\sigma = \frac{1}{\sqrt{T}}. \tag{28}$$

Geometric Brownian motion model

By use of the Bayes formula we find that the conditional probability density is of the Gaussian form:

$$\pi_t(x) = \sqrt{\frac{T}{2\pi(T-t)}} \exp\left(-\frac{1}{2(T-t)}\left(\sqrt{T}x - \xi_t\right)^2\right), \tag{29}$$

and has the following dynamics

$$\mathrm{d}\pi_t(x) = \frac{1}{T-t}\left(\sqrt{T}x - \xi_t\right)\pi_t(x)\mathrm{d}\xi_t. \tag{30}$$

A short calculation then shows that the value of the asset at time $t < T$ is

$$\begin{aligned} S_t &= \mathrm{e}^{-r(T-t)}\mathbb{E}_t[D_T] \\ &= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty} S_0\mathrm{e}^{rT+\nu\sqrt{T}x-\frac{1}{2}\nu^2 T}\pi_t(x)\mathrm{d}x \\ &= S_0 \exp\left(rt + \nu\xi_t - \tfrac{1}{2}\nu^2 t\right). \end{aligned} \tag{31}$$

The surprising fact in this example is that $\{\xi_t\}$ itself turns out to be the innovation process.

Indeed, it is not too difficult to verify that $\{\xi_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

Hence, setting $W_t = \xi_t$ for $0 \leq t < T$ we obtain the standard geometric Brownian motion model:

$$S_t = S_0 \exp\left(rt + \nu W_t - \tfrac{1}{2}\nu^2 t\right). \tag{32}$$

We see therefore that starting with an information process of the form (27) we are able to recover the familiar asset price dynamics given by (32).

Orthogonal decomposition of Brownian motion

An important point to note here is that the Brownian bridge process $\{\beta_{tT}\}$ appears quite naturally in this context.

In fact, if we start with (32) then we can make use of the following orthogonal decomposition of the Brownian motion:

$$W_t = \frac{t}{T} W_T + \left(W_t - \frac{t}{T} W_T \right). \tag{33}$$

The second term in the right, independent of the first term on the right, is a standard representation for a Brownian bridge process:

$$\beta_{tT} = W_t - \frac{t}{T} W_T. \tag{34}$$

Thus by writing

$$X_T = W_T/\sqrt{T} \quad \text{and} \quad \sigma = 1/\sqrt{T} \tag{35}$$

we find that the right side of (33) is indeed the market information.

In other words, formulated in the information-based framework, the standard Black-Scholes-Merton theory can be expressed in terms of a normally distributed X-factor and an independent Brownian bridge noise process.

The special feature of the Black-Scholes theory therefore is the fact that the information flow rate takes the specific form $\sigma = 1/\sqrt{T}$.

3. Application to credit risk management

Credit derivatives: an information-based approach

It turns out that the information-based approach has a nice application in modelling credit risk.

Models for credit risk management and for the pricing of credit derivatives are usually classified into two types: structural and reduced-form.

The latter are more commonly used in practice on account of their tractability, and the fact that fewer assumptions are required about the nature of the debt obligations involved and the circumstances that might lead to default.

Most reduced-form models are based on the introduction of a random time of default, where the default time is modelled as the time at which the integral of a random intensity process first hits a certain random critical level.

An unsatisfactory feature of intensity models is that they do not adequately take into account the fact that defaults are typically associated with a failure in the delivery of a promised cash-flow—for example, a missed coupon payment.

Another drawback of the intensity approach is that it is not well adapted to the situation where one wants to model the rise and fall of credit spreads.

The present framework provides a way forward in dealing with these issues.

The credit model in more detail

Let us consider the case of a simple credit-risky discount bond that matures at time T to pay a principal of h_1 dollars, providing there is no default.

In the event of default, the bond pays h_0 dollars.

For example we might consider the case $h_1 = 1$ and $h_0 = 0$.

When just two such payoffs are possible we shall call the resulting structure a 'binary' discount bond.

Let us write p_1 for the probability that the bond will pay h_1, and p_0 for the probability that the bond will pay h_0.

The probabilities here are the risk-neutral probabilities, and hence build in any risk adjustments required in expectations needed in order to deduce prices.

If we write B_{0T} for the price at time 0 of the risky discount bond then

$$B_{0T} = P_{0T}(p_1 h_1 + p_0 h_0). \tag{36}$$

It follows that, providing we know the market data B_{0T} and P_{0T}, we can infer the *a priori* probabilities p_1 and p_0:

$$p_0 = \frac{1}{h_1 - h_0}\left(h_1 - \frac{B_{0T}}{P_{0T}}\right), \quad p_1 = \frac{1}{h_1 - h_0}\left(\frac{B_{0T}}{P_{0T}} - h_0\right). \tag{37}$$

Let H_T denotes the random payout of the bond.

The true value of H_T, therefore, is not fully accessible until time T.

That is to say, we assume that H_T is $\mathcal{F}_T$-measurable, but is not necessarily $\mathcal{F}_t$-measurable for $t < T$.

Modelling the flow of information

We make the reasonable assumption that *some* partial information regarding the value of the principal repayment H_T is available at earlier times.

More precisely, we shall assume that the following $\{\mathcal{F}_t\}$-adapted *market* *information process* is accessible to market participants:

$$\xi_t = \sigma H_T t + \beta_{tT}. \tag{38}$$

Here the process $\{\beta_{tT}\}_{0\le t\le T}$ is a standard Brownian bridge over $[0, T]$, independent of H_T, and thus represents "pure noise".

Expression for the price process of a credit-risky bond

For simplicity, we assume that the only information available about H_T at times earlier than T comes from observations of $\{\xi_t\}$.

More specifically, if we denote by $\mathcal{F}_t^{\xi} \subset \mathcal{F}_t$ the subalgebra of $\mathcal{F}_t$ generated by $\{\xi_t\}_{0\le s\le t}$, then our assumption is that

$$\mathbb{E}\left[H_T | \mathcal{F}_t\right] = \mathbb{E}\left[H_T \middle| \mathcal{F}_t^{\xi}\right]. \tag{39}$$

Now we are in a position to determine the price-process $\{B_{tT}\}_{0\le t\le T}$ for a credit-risky discount bond with payout H_T.

In particular, we wish to calculate

$$B_{tT} = P_{tT}\mathbb{E}\left[H_T \middle| \mathcal{F}_t^{\xi}\right]. \tag{40}$$

From the result obtained earlier we see that the conditional expectation

$$H_{tT} = \mathbb{E}\left[H_T \middle| \mathcal{F}_t^{\xi} \right] \tag{41}$$

can be worked out explicitly.

The result is given by the following expression:

$$H_{tT} = \frac{p_0 h_0 \exp\left[\frac{T}{T-t}\left(\sigma h_0 \xi_t - \frac{1}{2}\sigma^2 h_0^2 t\right)\right] + p_1 h_1 \exp\left[\frac{T}{T-t}\left(\sigma h_1 \xi_t - \frac{1}{2}\sigma^2 h_1^2 t\right)\right]}{p_0 \exp\left[\frac{T}{T-t}\left(\sigma h_0 \xi_t - \frac{1}{2}\sigma^2 h_0^2 t\right)\right] + p_1 \exp\left[\frac{T}{T-t}\left(\sigma h_1 \xi_t - \frac{1}{2}\sigma^2 h_1^2 t\right)\right]}. \tag{42}$$

The value of the bond at any time t before maturity can be expressed as a function of the value of ξ_t.

Since $\{\xi_t\}$ is given by a combination of the random bond payoff and an independent Brownian bridge, this means that it is straightforward to *simulate* sample trajectories for the bond price process.

The bond price trajectories will then rise and fall randomly in line with the fluctuating information about the likely final payoff.

Determining the conditional expectation

Now consider the more general situation where the discount bond pays out the value $H_T = h_i$ $(i = 0, 1, \ldots, n)$ with *a priori* probability $\mathbb{Q}[H_T = h_i] = p_i$.

For convenience we assume $h_n > h_{n-1} > \cdots > h_1 > h_0$.

The case $n = 1$ then corresponds to the binary bond we have just considered.

We think of $H_T = h_n$ as the case of no default, and all of the other cases as various possible degrees of partial recovery.

Writing $\xi_t = \sigma t H_T + \beta_{tT}$ as before, we want to find the conditional expectation $\mathbb{E}[H_T|\mathcal{F}_t^{\xi}]$ of the bond payoff.

It follows from the Markovian property of $\{\xi_t\}$ that the conditioning with respect to the σ-algebra $\mathcal{F}_t^{\xi}$ can be replaced by conditioning with respect to ξ_t.

Therefore, writing

$$H_{tT} = \mathbb{E}[H_T|\xi_t] \tag{43}$$

for the conditional expectation of H_T given ξ_t, we have

$$H_{tT} = \sum_i h_i \pi_{it}. \tag{44}$$

The process $\{\pi_{it}\}$ is defined by

$$\pi_{it} = \mathbb{Q}(H_T = h_i|\xi_t). \tag{45}$$

Thus $\{\pi_{it}\}$ is the conditional probability that the credit-risky bond pays out h_i.

The *a priori* probability p_i and the *a posteriori* probability π_{it} at time t are related by the Bayes formula:

$$\mathbb{Q}(H_T = h_i|\xi_t) = \frac{p_i \rho(\xi_t|H_T = h_i)}{\sum_i p_i \rho(\xi_t|H_T = h_i)}. \tag{46}$$

Here $\rho(\xi|H_T = h_i)$ is the conditional density function for the random variable ξ_t, given by

$$\rho(\xi|H_T = h_i) = \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{T(\xi - \sigma t h_i)^2}{2t(T-t)}\right). \tag{47}$$

Expression (47) can be deduced from the fact that conditional on $H_T = h_i$ the random variable ξ_t is normally distributed with mean $\sigma t h_i$ and variance $t(T-t)/T$.

As a consequence of (46) and (47), we deduce that the conditional probabilities are

$$\pi_{it} = \frac{p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}. \tag{48}$$

It follows that

$$H_{tT} = \frac{\sum_i p_i h_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}. \tag{49}$$

This is the desired expression for the conditional expectation of the bond payoff.

The discount-bond price process $\{B_{tT}\}$ is therefore given by

$$B_{tT} = P_{tT} H_{tT}. \tag{50}$$

Defaultable discount bond dynamics

Let us now proceed to analyse the dynamics of the discount bond price process.

The key relation we need for determining the dynamics of the bond price process is that the conditional probability $\{\pi_{it}\}$ satisfies a diffusion equation of the form

$$\mathrm{d}\pi_{it} = \frac{\sigma T}{T-t}(h_i - H_{tT})\pi_{it}\mathrm{d}W_t. \tag{51}$$

The process $\{W_t\}_{0\leq t<T}$ appearing here, defined by

$$W_t = \xi_t + \int_0^t \frac{1}{T-s}\,\xi_s\,\mathrm{d}s - \sigma T \int_0^t \frac{1}{T-s}\,H_{sT}\,\mathrm{d}s, \tag{52}$$

is an $\{\mathcal{F}_t\}$-Brownian motion.

The fact that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion is a highly nontrivial result, and can be verified directly by showing that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-martingale and that $(\mathrm{d}W_t)^2 = \mathrm{d}t$.

The Brownian motion $\{W_t\}$ arising in this way can thus be regarded as part of the information accessible to market participants.

We note that, unlike $\{\beta_{tT}\}$, the value of W_t contains some "real" information relevant to the ultimate fate of the bond payoff.

As before, we call $\{W_t\}$ the "innovation process".

It follows from (44) and (51) that for the discount bond dynamics we have

$$\mathrm{d}B_{tT} = r_t B_{tT}\,\mathrm{d}t + \Sigma_{tT}\,\mathrm{d}W_t. \tag{53}$$

Here $r_t = -\partial \ln P_{0t}/\partial t$ is the deterministic short rate.

The absolute bond volatility Σ_{tT} is given by

$$\Sigma_{tT} = \frac{\sigma T}{T-t} P_{tT} V_{tT}. \tag{54}$$

The process $\{V_{tT}\}$ appearing here is defined by the relation

$$V_{tT} = \sum_i (h_i - H_{tT})^2 \pi_{it}. \tag{55}$$

We see that V_{tT} has the interpretation of being the conditional variance of the terminal payoff H_T.

It should be apparent that as the maturity date is approached the absolute discount bond volatility will be high unless the conditional probability has most of its mass concentrated around the "true" outcome.

Simulation of credit-risky bond price processes

The present framework allows for a very simple and natural simulation methodology for the dynamics of defaultable bonds and related structure.

In the case of a defaultable discount bond all we need to do is to simulate the dynamics of the process $\{\xi_t\}$.

Thus for each "run" of the simulation we choose at random a value for H_T (in accordance with the correct *a priori* probabilities), and a sample path for the Brownian bridge.

That is to say, each simulation corresponds to a choice of $\omega \in \Omega$, and for each such choice we simulate the path

$$\xi_t(\omega) = \sigma t H_T(\omega) + \beta_{tT}(\omega) \tag{56}$$

for $t \in [0, T]$.

One way to simulate a Brownian bridge is to write

$$\beta_{tT} = B_t - \frac{t}{T} B_T. \tag{57}$$

Here $\{B_t\}$ is a standard Brownian motion.

It is straightforward to verify that if $\{\beta_{tT}\}$ is defined in this way then it has the correct auto-covariance properties.

Since the bond price at time t is expressed directly as a function of ξ_t, this means that a pathwise simulation of the trajectory of the bond price is feasible for an arbitrary number of recovery levels.

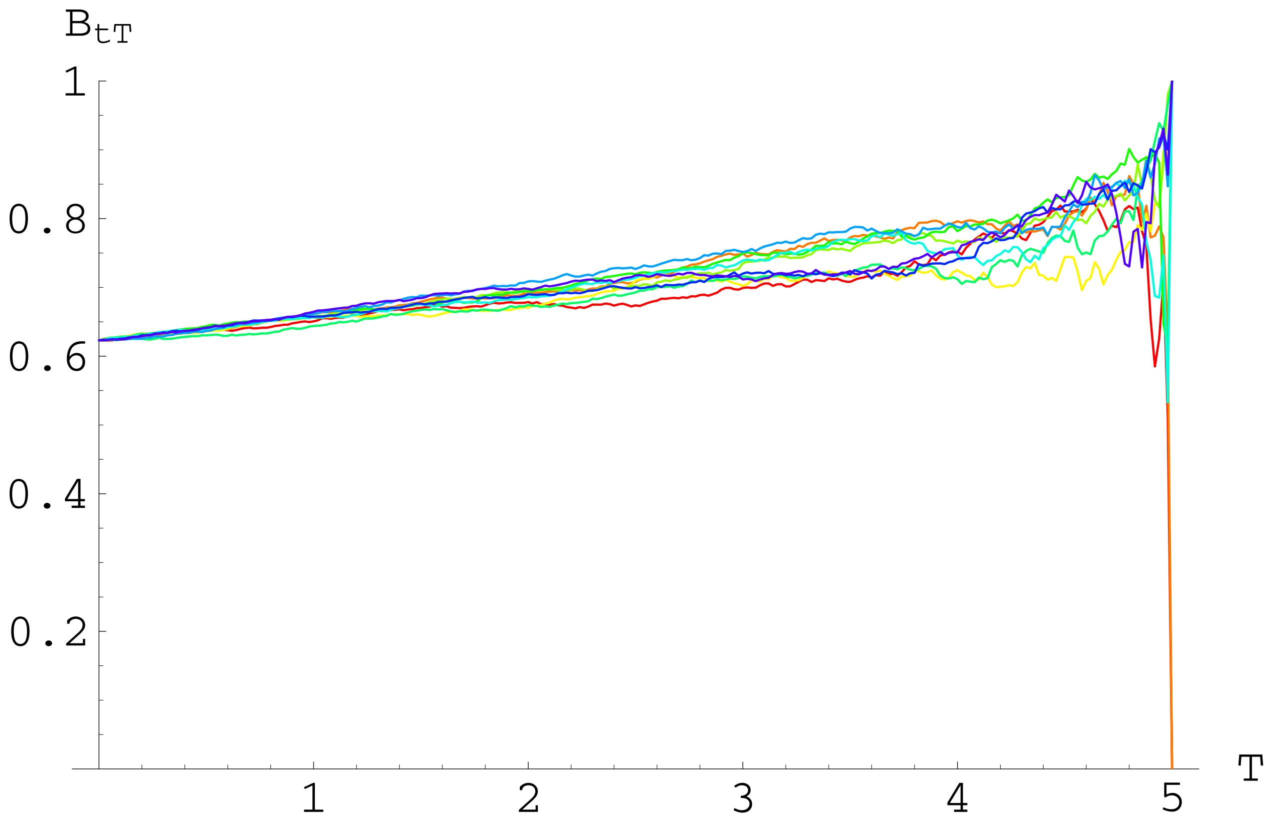

Figure 2: Examples of sample paths for $\sigma = 0.04$.

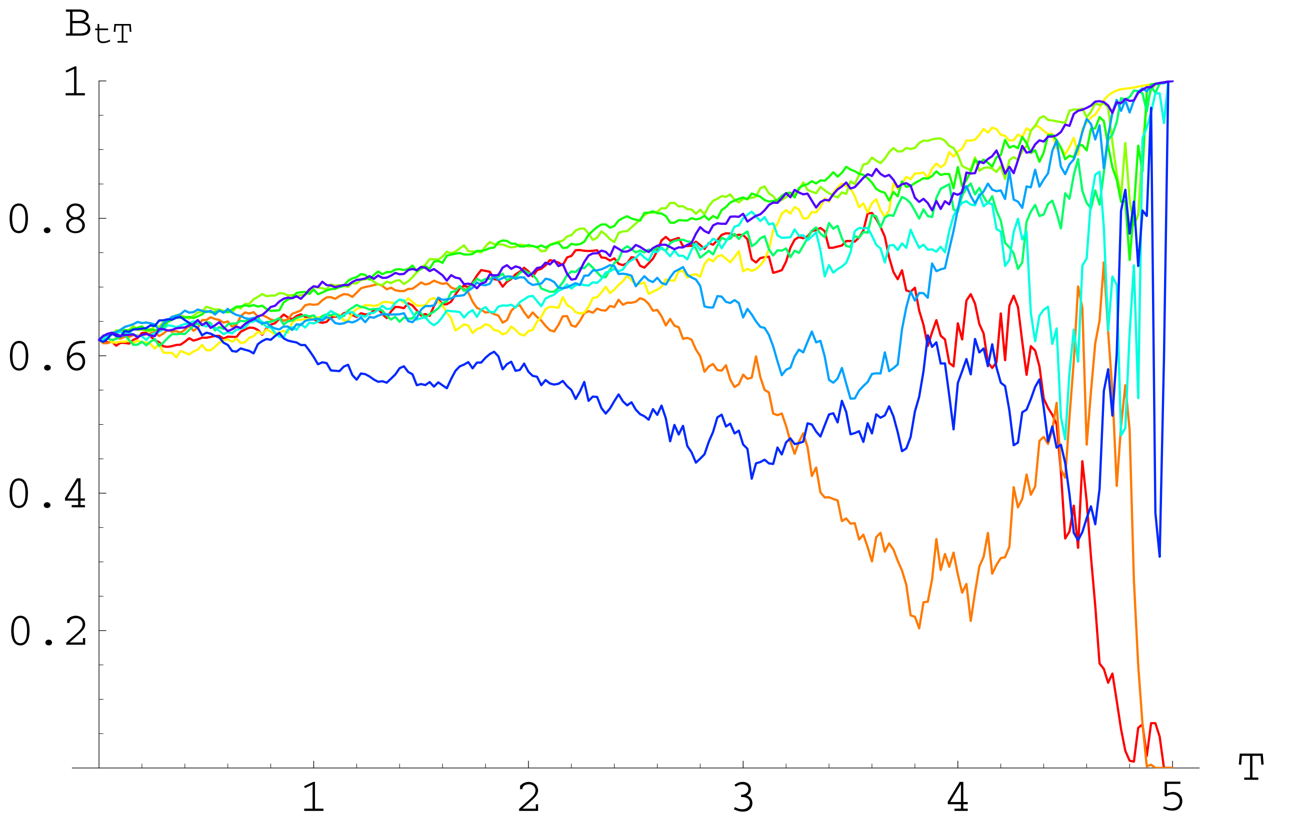

Figure 3: Examples of sample paths for $\sigma = 0.2$.

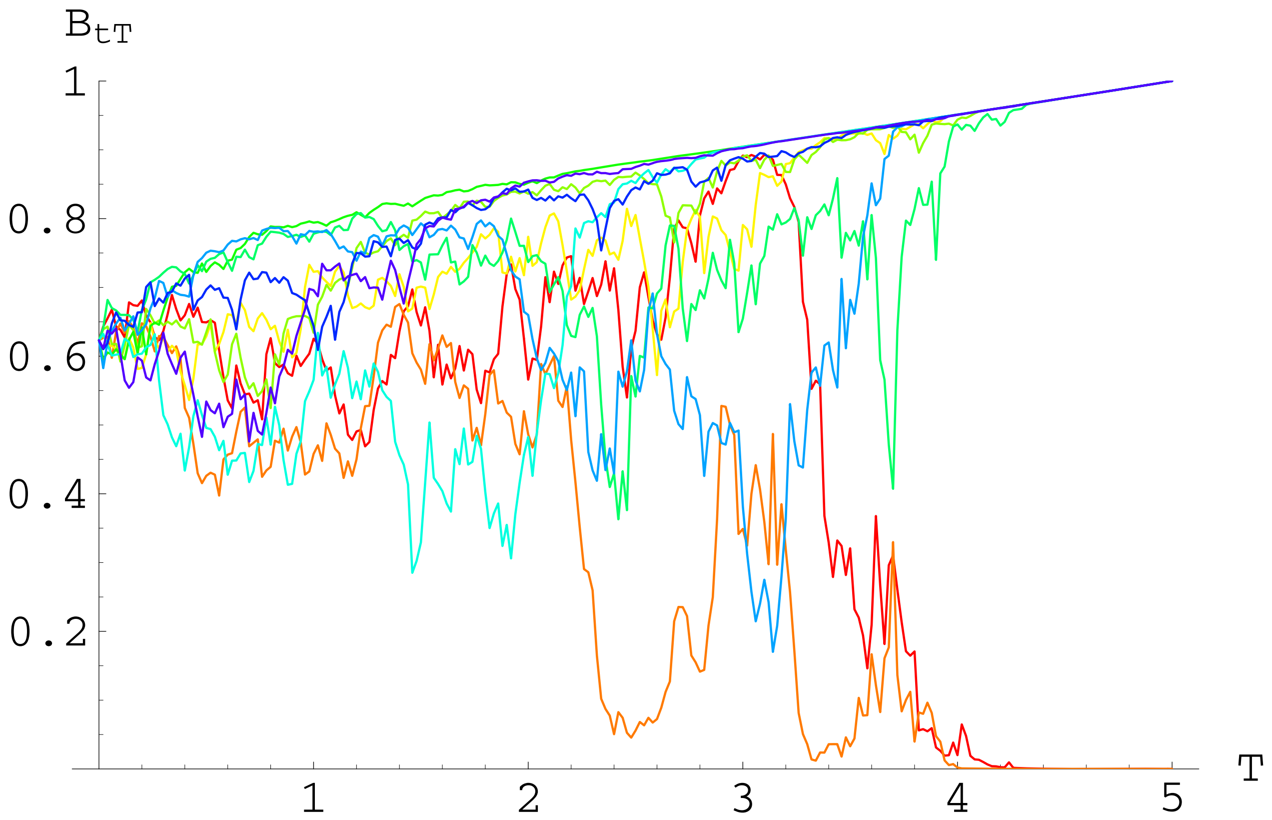

Figure 4: Examples of sample paths for $\sigma = 1$.

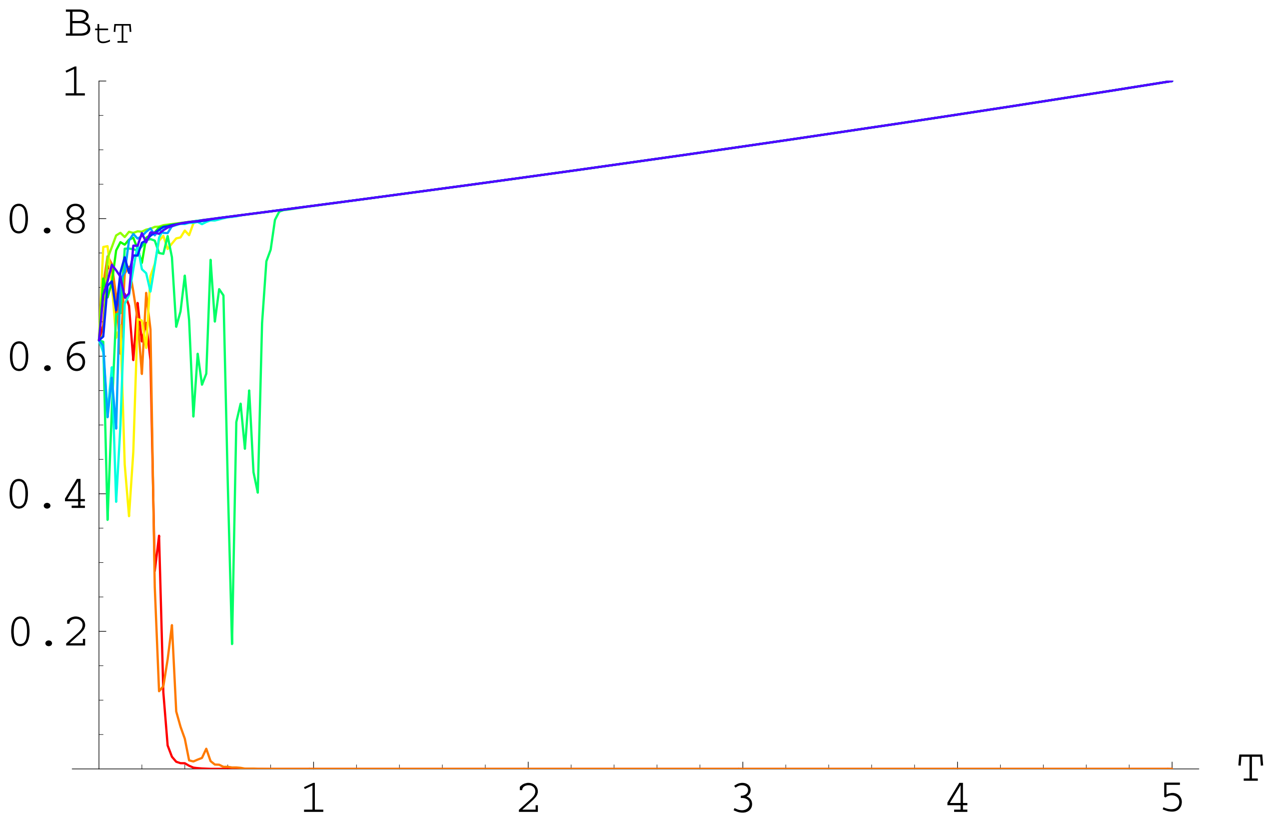

Figure 5: Examples of sample paths for $\sigma = 5$.

Dynamic consistency and model calibration

The information-based framework exhibits a property that might appropriately be called "dynamic consistency".

Loosely speaking, the question is: if the information process is given as described, but then we re-initialise the model at some specified intermediate time, is it still the case that the dynamics of the model moving forward from that intermediate time can be consistently represented by an information process?

To answer this question we proceed as follows.

First we define a standard Brownian bridge over the interval $[t, T]$ to be a Gaussian process $\{\gamma_{uT}\}_{t\leq u\leq T}$ satisfying $\gamma_{tT} = 0$, $\gamma_{TT} = 0$, $\mathbb{E}[\gamma_{uT}] = 0$ for all $u \in [t, T]$, and

$$\mathbb{E}[\gamma_{uT}\gamma_{vT}] = (u - t)(T - v)/(T - t) \tag{58}$$

for all u, v such that $t \leq u \leq v \leq T$.

Let $\{\beta_{tT}\}_{0\leq t\leq T}$ be a standard Brownian bridge over the interval $[0, T]$, and *define the process* $\{\gamma_{uT}\}_{t\le u\le T}$ *by*

$$\gamma_{uT} = \beta_{uT} - \frac{T-u}{T-t}\,\beta_{tT}. \tag{59}$$

Then $\{\gamma_{uT}\}_{t\le u\le T}$ *is a standard Brownian bridge over the interval* $[t,T]$*, and is independent of* $\{\beta_{sT}\}_{0\le s\le t}$.

Now let the information process $\{\xi_s\}_{0\le s\le T}$ be given, and fix an intermediate time $t \in (0,T)$.

Then for all $u \in [t,T]$ let us define a process $\{\eta_u\}$ by

$$\eta_u = \xi_u - \frac{T-u}{T-t}\,\xi_t. \tag{60}$$

We claim that $\{\eta_u\}$ is an information process over the time interval $[t,T]$.

In fact, a short calculation establishes that

$$\eta_u = \tilde{\sigma} H_T (u-t) + \gamma_{uT}, \tag{61}$$

where $\{\gamma_{uT}\}_{t\le u\le T}$ is a standard Brownian bridge over the interval $[t,T]$, independent of H_T, and $\tilde{\sigma} = \sigma T/(T-t)$.

Thus the "original" information process proceeds from time 0 up to time t.

At that time we can re-calibrate the model by taking note of the value of the random variable ξ_t, and introducing the re-initialised information process $\{\eta_u\}$.

The new information process depends on H_T; but since the value of ξ_t is supplied, the *a priori* probability distribution for H_T now changes to the appropriate *a posteriori* distribution consistent with the knowledge of ξ_t.

These interpretive remarks can be put into a more precise form as follows.

For $0 \leq t \leq u < T$ we want a formula for the conditional probability π_{iu} expressed in terms of the information η_u and the "new" *a priori* probability π_{it}.

Such a formula exists, and is given as follows:

$$\pi_{iu} = \frac{\pi_{it} \exp\left[\frac{T-t}{T-u}\left(\tilde{\sigma} h_i \eta_u - \frac{1}{2}\tilde{\sigma}^2 h_i^2 (u-t)\right)\right]}{\sum_i \pi_{it} \exp\left[\frac{T-t}{T-u}\left(\tilde{\sigma} h_i \eta_u - \frac{1}{2}\tilde{\sigma}^2 h_i^2 (u-t)\right)\right]}. \tag{62}$$

This remarkable relation can be verified by substituting the given expressions for π_{it}, η_u, and $\tilde{\sigma}$ into the right-hand side of (62).

But (62) has the same structure as the original formula for π_{it}, and thus we see that the model exhibits *manifest dynamic consistency*.

4. Pricing and hedging credit derivatives

Options on credit-risky bonds

We now turn to consider the pricing of options on credit-risky bonds.

In the case of a binary bond there is an exact solution for the valuation of European-style vanilla options.

The resulting expression for the option price exhibits a structure that is strikingly analogous to that of the Black-Scholes option pricing formula.

We consider the value at time 0 of an option that is exercisable at a fixed time $t > 0$ on a credit-risky discount bond that matures at time $T > t$.

The value C_0 of a call option is

$$C_0 = P_{0t}\mathbb{E}\left[(B_{tT} - K)^+\right], \tag{63}$$

where K is the strike price.

Inserting formulae (49) and (50) for B_{tT} into the valuation formula (63) for the option, we obtain

$$
\begin{aligned}
C_0 &= P_{0t}\,\mathbb{E}\left[(P_{tT}H_{tT}-K)^+\right] \\
&= P_{0t}\,\mathbb{E}\left[\left(\sum_{i=0}^{n} P_{tT}\pi_{it}h_i - K\right)^+\right] \\
&= P_{0t}\,\mathbb{E}\left[\left(\frac{1}{\Phi_t}\sum_{i=0}^{n} P_{tT}p_{it}h_i - K\right)^+\right] \\
&= P_{0t}\,\mathbb{E}\left[\frac{1}{\Phi_t}\left(\sum_{i=0}^{n}\Big(P_{tT}h_i - K\Big)p_{it}\right)^+\right].
\end{aligned}
\tag{64}
$$

Here the random variables p_{it}, $i = 0, 1, \ldots, n$, are the "unnormalised" conditional probabilities, defined by

$$
p_{it} = p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \tfrac{1}{2}\sigma^2 h_i^2 t\right)\right].
\tag{65}
$$

Then $\pi_{it} = p_{it}/\Phi_t$, where $\Phi_t = \sum_i p_{it}$, or, more explicitly,

$$\Phi_t = \sum_{i=0}^{n} p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \tfrac{1}{2}\sigma^2 h_i^2 t\right)\right]. \tag{66}$$

Change of measure technique

Our plan now is to use the factor $1/\Phi_t$ appearing in (64) to make a change of probability measure on $(\Omega, \mathcal{F}_t)$.

To this end, we fix a time horizon u at or beyond the option expiration but before the bond maturity, so $t \leq u < T$.

We define a process $\{\Phi_t\}_{0\leq t\leq u}$ by use of the expression (66), where now we let t vary in the range $[0, u]$.

It is a straightforward exercise in Ito calculus, making use of (52), to verify that

$$\mathrm{d}\Phi_t = \sigma^2\left(\frac{T}{T-t}\right)^2 H_{tT}^2 \Phi_t\,\mathrm{d}t + \sigma\frac{T}{T-t}H_{tT}\Phi_t\,\mathrm{d}W_t. \tag{67}$$

It follows then that

$$\mathrm{d}\Phi_t^{-1} = -\sigma \frac{T}{T-t} H_{tT} \Phi_t^{-1} \mathrm{d}W_t, \tag{68}$$

and hence that

$$\Phi_t^{-1} = \exp\left(-\sigma \int_0^t \frac{T}{T-s} H_{sT}\, \mathrm{d}W_s - \tfrac{1}{2}\sigma^2 \int_0^t \frac{T^2}{(T-s)^2} H_{sT}^2\, \mathrm{d}s \right). \tag{69}$$

Since $\{H_{sT}\}$ is bounded, and $s \leq u < T$, we see that the process $\{\Phi_s^{-1}\}_{0\leq s\leq u}$ is a martingale.

In particular, since $\Phi_0 = 1$, we deduce that $\mathbb{E}^{\mathbb{Q}}[\Phi_t^{-1}] = 1$, where t is the option maturity date, and hence that the factor $1/\Phi_t$ in (64) can be used to effect a change of measure on $(\Omega, \mathcal{F}_t)$.

Writing $\mathbb{B}_T$ for the new probability measure thus defined, we have

$$C_0 = P_{0t}\, \mathbb{E}^{\mathbb{B}_T} \left[\left(\sum_{i=0}^{n} \Big(P_{tT} h_i - K \Big) p_{it} \right)^+ \right]. \tag{70}$$

We call $\mathbb{B}_T$ the "bridge" measure because it has the special property that it makes $\{\xi_s\}_{0\leq s\leq t}$ a $\mathbb{B}_T$-Gaussian process with mean zero and covariance $\mathbb{E}^{\mathbb{B}_T}[\xi_r \xi_s] = r(T-s)/T$ for $0 \leq r \leq s \leq t$.

In other words, with respect to the measure $\mathbb{B}_T$, and over the interval $[0, t]$, the information process has the law of a standard Brownian bridge over the interval $[0, T]$.

The bridge measure

The proof that $\{\xi_s\}_{0\leq s\leq t}$ has the claimed properties under the measure $\mathbb{B}_T$ is as follows.

For convenience we introduce a process $\{W_t^*\}_{0\leq t\leq u}$ which we define as the following Brownian motion with drift in the $\mathbb{Q}$-measure:

$$W_t^* = W_t + \sigma \int_0^t \frac{T}{T-s} H_{sT}\, \mathrm{d}s. \tag{71}$$

It is straightforward to check that on $(\Omega, \mathcal{F}_t)$ the process $\{W_t^*\}_{0\leq t\leq u}$ is a Brownian motion with respect to the measure defined by use of the density martingale $\{\Phi_t^{-1}\}_{0\leq t\leq u}$ given by (69).

It then follows from the definition of $\{W_t\}$, given in (52), that

$$W_t^* = \xi_t + \int_0^t \frac{1}{T-s} \xi_s\, \mathrm{d}s. \tag{72}$$

Taking the stochastic differential of each side of this relation, we deduce that

$$d\xi_t = -\frac{1}{T-t}\,\xi_t\,dt + dW^*_t. \tag{73}$$

We note, however, that (73) is the stochastic differential equation satisfied by a Brownian bridge over the interval $[0,T]$.

Thus we see that in the measure $\mathbb{B}_T$ defined on $(\Omega,\mathcal{F}_t)$ the process $\{\xi_s\}_{0\le s\le t}$ has the properties of a standard Brownian bridge over $[0,T]$, albeit restricted to the interval $[0,t]$.

For the transformation back from $\mathbb{B}_T$ to $\mathbb{Q}$ on $(\Omega,\mathcal{F}_t)$, the appropriate density martingale $\{\Phi_t\}_{0\le t\le u}$ with respect to $\mathbb{B}_T$ is given by:

$$\Phi_t = \exp\left(\sigma\int_0^t \frac{T}{T-s}\,H_{sT}\,dW^*_s - \tfrac{1}{2}\sigma^2\int_0^t \frac{T^2}{(T-s)^2}\,H^2_{sT}\,ds\right). \tag{74}$$

The crucial point that follows is that the random variable ξ_t is $\mathbb{B}_T$-Gaussian.

Options on binary bonds

In the case of a binary discount bond, therefore, the relevant expectation for determining the option price can be carried out by standard techniques, and we are led to a formula of the Black-Scholes type.

In particular, for a binary bond, Equation (70) reads

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left((P_{tT}h_1 - K)p_{1t} + (P_{tT}h_0 - K)p_{0t}\right)^+\right], \tag{75}$$

where p_{0t} and p_{1t} are given by

$$\begin{aligned} p_{0t} &= p_0 \exp\left[\tfrac{T}{T-t}\left(\sigma h_0 \xi_t - \tfrac{1}{2}\sigma^2 h_0^2 t\right)\right], \\ p_{1t} &= p_1 \exp\left[\tfrac{T}{T-t}\left(\sigma h_1 \xi_t - \tfrac{1}{2}\sigma^2 h_1^2 t\right)\right]. \end{aligned} \tag{76}$$

To compute the value of (75) there are essentially three different cases that have to be considered: (i) $P_{tT}h_1 > P_{tT}h_0 > K$, (ii) $K > P_{tT}h_1 > P_{tT}h_0$, and (iii) $P_{tT}h_1 > K > P_{tT}h_0$.

In case (i) the option is certain to expire in the money.

Thus, making use of the fact that ξ_t is $\mathbb{B}_T$-Gaussian with mean zero and variance $t(T-t)/T$, we see that $\mathbb{E}^{\mathbb{B}_T}[p_{it}] = p_i$; hence in case (i) we have $C_0 = B_{0T} - P_{0t}K$.

In case (ii) the option expires out of the money, and thus $C_0 = 0$.

In case (iii) the option can expire in or out of the money, and there is a "critical" value of ξ_t above which the argument of (75) is positive.

This is obtained by setting the argument of (75) to zero and solving for ξ_t.

Writing $\bar{\xi}_t$ for the critical value, we find that $\bar{\xi}_t$ is determined by the relation

$$\frac{T}{T-t}\sigma(h_1 - h_0)\bar{\xi}_t = \ln\left[\frac{p_0(P_{tT}h_0 - K)}{p_1(K - P_{tT}h_1)}\right] + \tfrac{1}{2}\sigma^2(h_1^2 - h_0^2)\tau, \qquad (77)$$

where $\tau = tT/(T-t)$.

Next we note that since ξ_t is $\mathbb{B}_T$-Gaussian with mean zero and variance $t(T-t)/T$, for the purpose of computing the expectation in (75) we can set $\xi_t = Z\sqrt{t(T-t)/T}$, where Z is $\mathbb{B}_T$-Gaussian with zero mean and unit variance.

Then writing $\bar{Z}$ for the corresponding critical value of Z, we obtain

$$\bar{Z} = \frac{\ln\left[\frac{p_0(P_{tT}h_0 - K)}{p_1(K - P_{tT}h_1)}\right] + \tfrac{1}{2}\sigma^2(h_1^2 - h_0^2)\tau}{\sigma\sqrt{\tau}(h_1 - h_0)}. \qquad (78)$$

With this expression at hand, we can work out the expectation in (75).

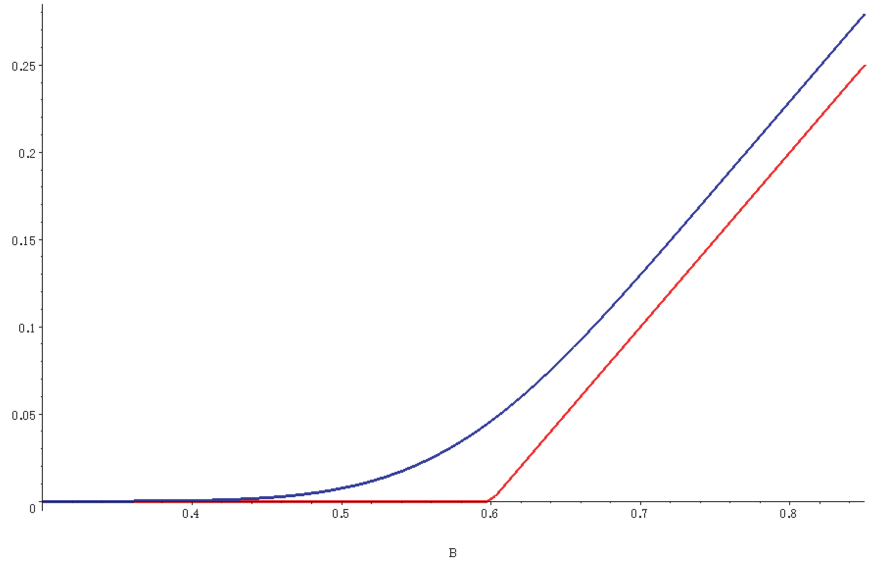

Figure 6: Call price as a function of initial bond price. The parameters are: $r = 5\%$, $\sigma = 25\%$, $T = 2$ year, $t = 1$ year, and $K = 0.6$.

We are thus led to the following option pricing formula:

$$C_0 = P_{0t}\Big[p_1(P_{tT}h_1 - K)N(d^+) - p_0(K - P_{tT}h_0)N(d^-)\Big]. \tag{79}$$

Here d^+ and d^- are defined by

$$d^{\pm} = \frac{\ln\left[\frac{p_1(P_{tT}h_1 - K)}{p_0(K - P_{tT}h_0)}\right] \pm \frac{1}{2}\sigma^2(h_1 - h_0)^2\tau}{\sigma\sqrt{\tau}(h_1 - h_0)}. \tag{80}$$

Greeks

A short calculation shows that the corresponding option delta, defined by $\Delta = \partial C_0/\partial B_{0T}$, is given by

$$\Delta = \frac{(P_{tT}h_1 - K)N(d^+) + (K - P_{tT}h_0)N(d^-)}{P_{tT}(h_1 - h_0)}. \tag{81}$$

This can be verified by using (79) to determine the dependency of the option price C_0 on the initial bond price B_{0T}.

It is interesting to note that the parameter σ plays a role like that of the volatility parameter in the Black-Scholes model.

The more rapidly information is "leaked out" about the true value of the bond repayment, the higher the volatility.

It is straightforward to verify that the option price has a positive vega, i.e. that C_0 is an increasing function of σ.

This means that we can use bond option prices (or, equivalently, caps and floors) to back out an implied value for σ, and hence to calibrate the model.

Writing $\mathcal{V} = \partial C_0/\partial\sigma$ for the corresponding option vega, we obtain the following

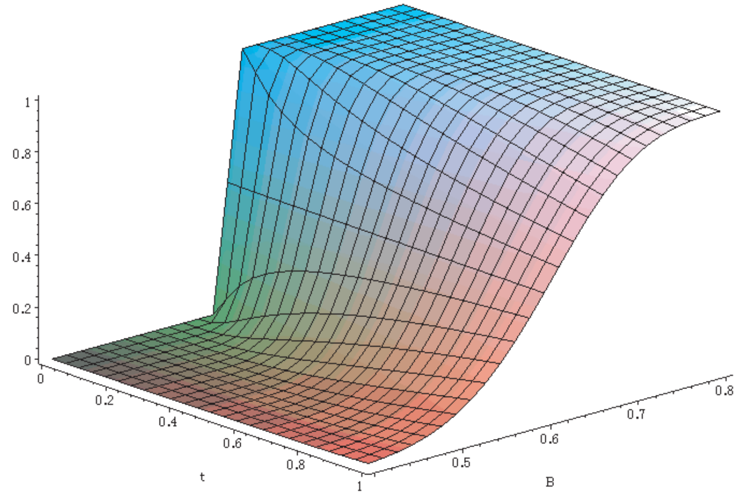

Figure 7: Delta hedge for a defaultable digital bond option with $r = 5\%$, $\sigma = 25\%$, $T = 2$ year, and $K = 0.6$.

positive expression:

$$\mathcal{V} = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-rt-\frac{1}{2}A}(h_1 - h_0)\sqrt{\tau p_0 p_1 (P_{tT}h_1 - K)(K - P_{tT}h_0)}, \tag{82}$$

where

$$A = \frac{1}{\sigma^2\tau(h_1-h_0)^2}\left(\ln\left[\frac{p_1(P_{tT}h_1 - K)}{p_0(K - P_{tT}h_0)}\right]\right)^2 + \tfrac{1}{4}\sigma^2\tau(h_1-h_0)^2. \tag{83}$$

Options on general bonds

In the more general case of a stochastic recovery, a semi-analytic option pricing formula can be obtained that, for practical purposes, can be regarded as fully tractable.

In particular, starting from (70) we consider the case where the strike price K lies in the range $P_{tT}h_{k+1} > K > P_{tT}h_k$ for some value of $k \in \{0, 1, \ldots, n\}$.

It is an exercise to verify that there exists a unique critical value of ξ_t such that the summation appearing in the argument of the $\max(x, 0)$ function in (70) vanishes.

Writing $\bar{\xi}_t$ for the critical value, which can be obtained by numerical methods, we define the scaled critical value $\bar{Z}$ as before, by setting $\bar{\xi}_t = \bar{Z}\sqrt{t(T-t)/T}$.

A calculation then shows that the option price is given by the following expression:

$$C_0 = P_{0t} \sum_{i=0}^{n} p_i \left(P_{tT}h_i - K\right) N(\sigma h_i \sqrt{\tau} - \bar{Z}). \tag{84}$$

5. Volatility and correlation: modelling dependent assets

Market factors and multiple cash flows

We proceed to consider the more general situation where the asset pays multiple dividends.

This will allow us to consider a wider range of financial instruments.

Let us write D_{T_k} $(k = 1, \ldots, n)$ for a set of random dividends paid at the pre-designated dates T_k $(k = 1, \ldots, n)$.

Possession of the asset at time t entitles the bearer to the cash flows occurring at times $T_k > t$.

For each value of k we introduce a set of independent random variables $X^{\alpha}_{T_k}$ $(\alpha = 1, \ldots, m_k)$, which we call market factors or X-factors.

For each value of α we assume that the market factor $X^{\alpha}_{T_k}$ is $\mathcal{F}_{T_k}$-measurable, where $\{\mathcal{F}_t\}$ is the market filtration.

For each value of k, the market factors $\{X^{\alpha}_{T_j}\}_{j \leq k}$ represent the independent elements that determine the cash flow occurring at time T_k.

Thus for each value of k the cash flow D_{T_k} is assumed to have the following structure:

$$D_{T_k} = \Delta_{T_k}(X^\alpha_{T_1}, X^\alpha_{T_2}, ..., X^\alpha_{T_k}), \tag{85}$$

where $\Delta_{T_k}(X^\alpha_{T_1}, X^\alpha_{T_2}, ..., X^\alpha_{T_k})$ is a function of $\sum_{j=1}^{k} m_j$ variables.

For each cash flow it is, so to speak, the job of the financial analyst (or actuary) to determine the relevant independent market factors, and the form of the cash-flow function Δ_{T_k} for each cash flow.

With each market factor $X^\alpha_{T_k}$ we associate an information process $\{\xi^\alpha_{tT_k}\}_{0\leq t\leq T_k}$ of the form

$$\xi^\alpha_{tT_k} = \sigma^\alpha_{T_k} X^\alpha_{T_k} t + \beta^\alpha_{tT_k}. \tag{86}$$

The X-factors and the Brownian bridge processes are all independent.

The parameter $\sigma^\alpha_{T_k}$ determines the rate at which the market factor $X^\alpha_{T_k}$ is revealed.

The market filtration $\{\mathcal{F}_t\}$ is generated by the totality of the independent information processes $\{\xi^{\alpha}_{tT_k}\}_{0\leq t\leq T_k}$ for $k=1,\ldots,n$ and $\alpha=1,\ldots,m_k$.

Hence, the price of the asset is given by

$$S_t = \sum_{k=1}^{n} \mathbb{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}_t \left[D_{T_k} \right] . \tag{87}$$

As an elementary example consider a two-factor bond price process, where the cash flow is given by

$$D_T = X_T^{(1)} X_T^{(2)} . \tag{88}$$

Assuming $X_T^{(1)}$, $X_T^{(2)}$ are binary random variables taking values $0, 1$, the resulting bond price sample paths are simulated for a range of values for σ_1, σ_2 (Figure 8).

Assets with common factors

The multiple-dividend asset pricing model can be extended in a natural way to the situation where two or more assets are being priced.

In this case we consider a collection of N assets with price processes $\{S_t^{(i)}\}_{i=1,2,\ldots,N}$.

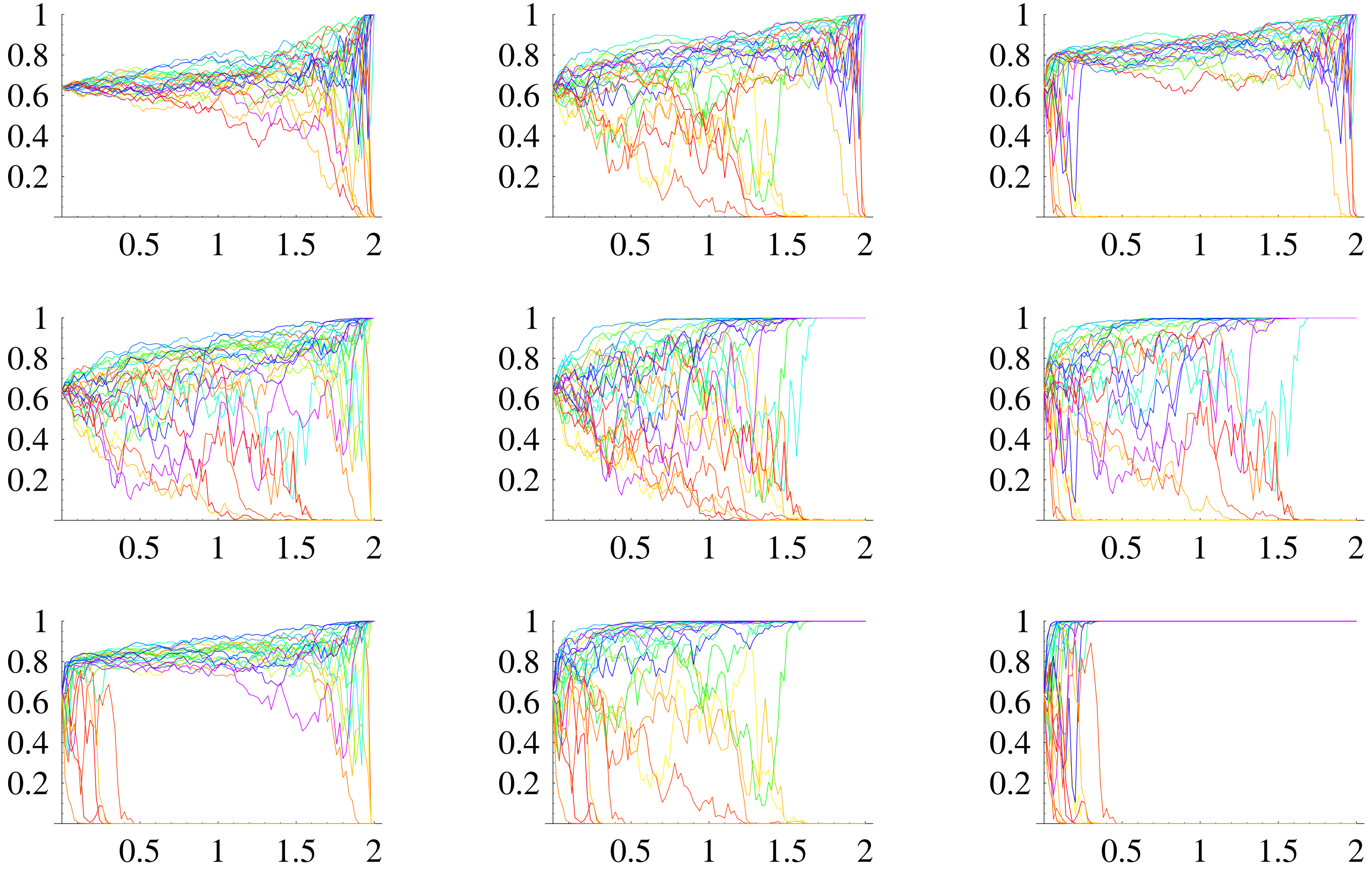

Figure 8: Sample paths for a two-factor bond price process, with $r = 0\%$.

With asset number (i) we associate the cash flows $\{D^{(i)}_{T_k}\}$ paid at the dates $\{T_k\}_{k=1,2,...,n}$.

The dates $\{T_k\}_{k=1,2,...,n}$ are not tied to any specific asset, but rather represent the totality of possible cash-flow dates of any of the given assets.

If a particular asset has no cash flow on one of the dates, then it is assigned a zero cash-flow for that date.

From this point, the theory proceeds exactly as in the single asset case.

That is to say, with each value of k we associate a set of X-factors $X^{\alpha}_{T_k}$ $(\alpha = 1, 2, \ldots, m_k)$, and a system of market information processes $\{\xi^{\alpha}_{tT_k}\}$.

The X-factors and the information processes are not tied to any particular asset.

The cash flow $D^{(i)}_{T_k}$ occurring at time T_k for asset number (i) is given by a cash flow function of the form

$$D^{(i)}_{T_k} = \Delta^{(i)}_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, ..., X^{\alpha}_{T_k}). \tag{89}$$

In other words, for each asset, each cash flow can depend on all of the X-factors that have been "activated" at that point.

Thus for the general multi-asset model we have the following price process:

$$S_t^{(i)} = \sum_{k=1}^{n} \mathbb{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}_t \left[D_{T_k}^{(i)} \right] . \tag{90}$$

It is possible in general for two or more assets to “share” an X-factor in association with one or more of the cash flows of each of the assets.

This in turn implies that the various assets will have at least one Brownian motion in common in the dynamics of their price processes.

We thus obtain a natural model for the correlation structures in the prices of these assets.

The intuition is that as new information comes in (whether “true” or “bogus”) there will be several different assets all affected by the news, and as a consequence there will be a correlated movement in their prices.

As a simple application we can ask how the price process of a T-maturity bond is affected if the same firm also issues a $T'(< T)$-maturity bond.

Several sample paths for the T-maturity bond price process in these scenarios are simulated in Figure 9.

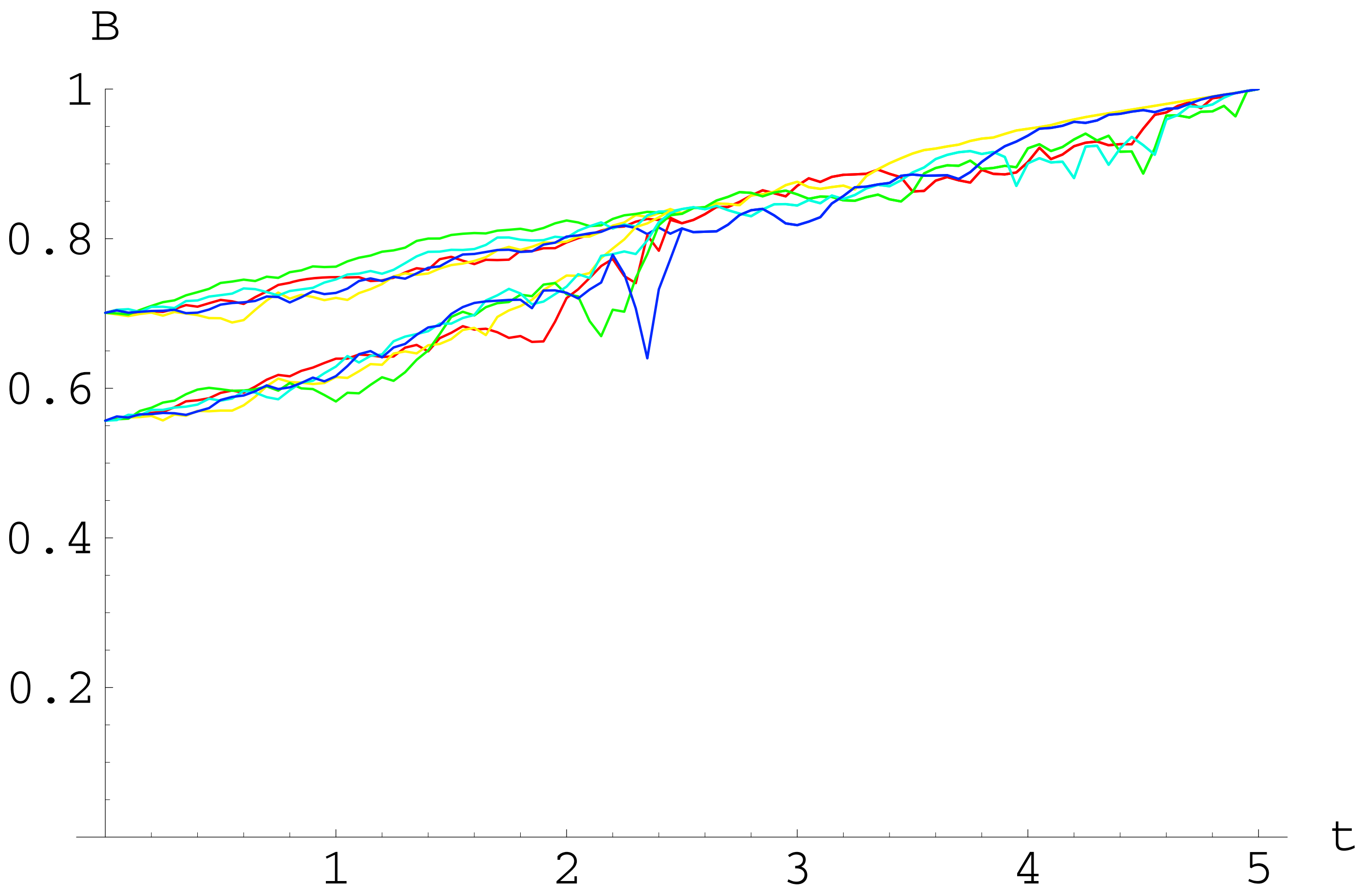

Figure 9: Sample paths for a T-maturity bond price with parameters $\sigma = 25\%$, $r = 5\%$, $T' = 2.5$, and $T = 5$. The cash flows are given by $H_{T'} = X$ and $H_T = XY$.

Origin of unhedgeable stochastic volatility

Based on the general model introduced here, we are in a position to make an observation concerning the nature of stochastic volatility in the equity markets.

The dynamics of the stochastic volatility can be derived without the need for any *ad hoc* assumptions.

In fact, a very specific dynamical model for stochastic volatility is obtained—thus leading to a possible means by which the theory proposed here might be tested.

We shall work out the volatility associated with the dynamics of the asset price process $\{S_t\}$ given by (87).

First, as an example, we consider the dynamics of an asset that pays a single dividend D_T at T.

We assume that the dividend depends on the market factors $\{X_T^\alpha\}_{\alpha=1,\ldots,m}$.

For $t < T$ we then have:

$$\begin{aligned} S_t &= P_{tT}\mathbb{E}^{\mathbb{Q}}\left[\Delta_T\left(X_T^1,\ldots,X_T^m\right)\Big|\,\xi_{tT}^1,\ldots,\xi_{tT}^m\right] \\ &= P_{tT}\int\cdots\int\Delta_T(x^1,\ldots,x^m)\,\pi_{tT}^1(x_1)\cdots\pi_{tT}^m(x_m)\,\mathrm{d}x_1\cdots\mathrm{d}x_m. \end{aligned} \quad (91)$$

Here the various conditional probability density functions $\pi_{tT}^{\alpha}(x)$ for $\alpha = 1,\ldots,m$ are

$$\pi_{tT}^{\alpha}(x) = \frac{p^{\alpha}(x)\exp\left[\frac{T}{T-t}\left(\sigma^{\alpha}\,x\,\xi_{tT}^{\alpha}-\frac{1}{2}(\sigma^{\alpha})^2\,x^2t\right)\right]}{\int_0^{\infty}p^{\alpha}(x)\exp\left[\frac{T}{T-t}\left(\sigma^{\alpha}\,x\,\xi_{tT}^{\alpha}-\frac{1}{2}(\sigma^{\alpha})^2\,x^2t\right)\right]\mathrm{d}x}, \quad (92)$$

where $p^{\alpha}(x)$ denotes the *a priori* probability density function for the factor X_T^{α}.

The drift of $\{S_t\}_{0\leq t<T}$ is given by the short rate.

This is because $\mathbb{Q}$ is the risk-neutral measure, and no dividend is paid before T.

Thus, we are left with the problem of determining the volatility of $\{S_t\}$.

We find that for $t < T$ the dynamical equation of $\{S_t\}$ assumes the form:

$$\mathrm{d}S_t = r_tS_t\mathrm{d}t + \sum_{\alpha=1}^{m}\Gamma_{tT}^{\alpha}\mathrm{d}W_t^{\alpha}. \quad (93)$$

Here the volatility term associated with factor number α is given by

$$\Gamma^{\alpha}_{tT} = \sigma^{\alpha}\frac{T}{T-t}P_{tT}\,\mathrm{Cov}\left[\Delta_T\left(X^1_T,\ldots,X^m_T\right),X^{\alpha}_T\middle|\,\mathcal{F}_t\right], \tag{94}$$

and $\{W^{\alpha}_t\}$ denotes the Brownian motion associated with the information process $\{\xi^{\alpha}_t\}$, as defined in (86).

The absolute volatility of $\{S_t\}$ is of the form

$$\Gamma_t = \left(\sum_{\alpha=1}^{m}\left(\Gamma^{\alpha}_{tT}\right)^2\right)^{1/2}. \tag{95}$$

For the dynamics of a multi-factor single-dividend paying asset we can thus write

$$\mathrm{d}S_t = r_tS_t\mathrm{d}t + \Gamma_t\mathrm{d}Z_t, \tag{96}$$

where the $\{\mathcal{F}_t\}$-Brownian motion $\{Z_t\}$ that drives the asset-price process is

$$Z_t = \int_0^t \frac{1}{\Gamma_s}\sum_{\alpha=1}^{m}\Gamma^{\alpha}_{sT}\,\mathrm{d}W^{\alpha}_s. \tag{97}$$

The point to note here is that in the case of a multi-factor model we obtain an unhedgeable stochastic volatility.

That is to say, although the asset price is in effect driven by a single Brownian motion, its volatility depends on a multiplicity of Brownian motions.

This means that in general an option position cannot be hedged with a position in the underlying asset.

The components of the volatility vector are given by the covariances of the cash flow and the independent market factors.

Unhedgeable stochastic volatility thus emerges from the multiplicity of uncertain elements in the market that affect the value of the future cash flow.

As a consequence we see that *in this framework we obtain a natural explanation for the origin of stochastic volatility*.

This result can be contrasted with, say, the Heston model, which despite its popularity suffers from the fact that it is *ad hoc* in nature.

Much the same can be said for the various generalisations of the Heston model used in commercial applications.

The approach to stochastic volatility proposed here is thus of a new character.

Expression (93) generalises to the case for which the asset pays a set of dividends D_{T_k} $(k = 1, \ldots, n)$, and for each k the dividend depends on the X-factors $\{\{X^{\alpha}_{T_j}\}^{\alpha=1,\ldots,m_j}_{j=1,\ldots,k}\}$.

The result can be summarised as follows:

The price process of a multi-dividend asset has the following dynamics:

$$
\begin{aligned}
dS_t \;=\; & r_t\, S_t\, dt + \sum_{k=1}^{n}\sum_{\alpha=1}^{m_k} \mathbb{1}_{\{t<T_k\}} \frac{\sigma^{\alpha}_{k} T_k}{T_k - t}\, P_{tT_k}\, \mathrm{Cov}\left[D_{T_k}, X^{\alpha}_{T_k} \middle| \mathcal{F}_t \right] dW^{\alpha k}_t \\
& + \sum_{k=1}^{n} D_{T_k} d\mathbb{1}_{\{t<T_k\}}, \qquad (98)
\end{aligned}
$$

where $D_{T_k} = \Delta_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, \cdots, X^{\alpha}_{T_k})$ *is the dividend at time* T_k $(k = 1, 2, \ldots, n)$.

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