

# Extended Libor Models and Their Calibration

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# Forward Libor Models

- **Tenor structure:**  $0 = T_0 < T_1 < \dots < T_M < T_{M+1}$  with accrual periods  $\delta_i := T_{i+1} - T_i$
- **Zero coupon bonds:**  $B_k(t)$ ,  $t \in [0, T_k]$  with  $B_k(T_k) = 1$
- **Forward Libor rates:**  $L_1(t), \dots, L_M(t)$

$$L_k(t) = \frac{1}{\delta_k} \left( \frac{B_k(t)}{B_{k+1}(t)} - 1 \right), \quad t \in [0, T_k], \quad k = 1, \dots, M$$

## Remark

$L_1, \dots, L_M$  are based on simple compounding that is an investor receives 1\$ at  $T_k$  and pays  $1 + \delta_k L_k(t)$  at  $T_{k+1}$

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# Libor Model under $P_{M+1}$

For  $i = 1, \dots, M$

$$\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \underbrace{\Gamma_i^\top d\mathcal{W}^{(M+1)}(t)}_{\text{Continuous Part}} + \underbrace{\int_E \psi_i(t, u) (\mu - \nu^{(M+1)})(dt, du)}_{\text{Jump Part}}$$

- $A_i$  are predictable drift processes
- $\mathcal{W}^{(M+1)}$  is a  $D$ -dimensional Brownian motion under  $P_{M+1}$
- $\Gamma_i$  are predictable  $D$ -dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$  is a random point measure on  $\mathbb{R}_+ \times E$  with  $P_{M+1}$ -compensator  $\nu^{(M+1)}(dt, du)$

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# Libor Model under $P_{M+1}$

We consider random point measures  $\mu$  of finite activity satisfying

$$\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],$$

where  $(s_l, u_l) \in \mathbb{R}_+ \times E$  are jumps of  $\mu$  and

- $N_t$  is a Poisson process with intensity  $\lambda$

- $E = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_m$

- $u_l \in \mathbb{R}^m$  is distributed with  $p_1(dx_1) \cdot \dots \cdot p_m(dx_m)$

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# Drift term under $P_{M+1}$

The requirement that  $L_j$  is a martingale under  $P^{(M+1)}$  implies

$$\begin{aligned}
 A_i^{(M+1)}(dt) = & - \sum_{j=i+1}^M \frac{\delta_j L_j}{1 + \delta_j L_j} \Gamma_i^\top \Gamma_j dt + \\
 & + \lambda(t) dt \int_{\mathbb{R}^m} \psi_i(u, t) p(du) \left[ \prod_{j=i+1}^M \left( 1 + \frac{\delta_j L_j - \psi_i(t, u)}{1 + \delta_j L_j} \right) \right].
 \end{aligned}$$

## Dynamic of $L_j$ under $P_{i+1}$

Since  $L_j$  is a martingale under  $P_{i+1}$

$$\frac{dL_j}{L_j} = \Gamma_i^\top dW^{(i+1)} + \int_E \psi_j(t, u)(\mu - \nu^{(i+1)})(dt, du),$$

where

$$dW^{(i+1)} = - \sum_{j=i+1}^M \frac{\delta_j L_{j-}}{1 + \delta_j L_{j-}} \Gamma_i dt + dW^{(M+1)}$$

is a standard Brownian motion under  $P_{i+1}$  and

$$\nu^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[ \prod_{j=i+1}^M \left( 1 + \frac{\delta_j L_{j-} \psi_j(t, u)}{1 + \delta_j L_{j-}} \right) \right].$$

## Dynamic of $L_i$ under $P_{i+1}$

The logarithmic version reads as

$$d \ln(L_i) = A^{(i+1)}(dt) + \Gamma_i^\top dW^{(i+1)} + d \left[ \sum_{i=1}^{N_t} \phi_i(s_l, u_l) \right]$$

with  $\phi_i = \ln(1 + \psi_i)$  and

$$A^{(i+1)}(dt) = -\frac{1}{2} |\Gamma_i|^2 dt - \int_{\mathbb{R}^m} \psi_i(t, u) \nu^{(i+1)}(dt, du)$$

### Observation

*For  $i < M$  the new compensator  $\nu^{(i+1)}$  is not deterministic and  $\ln(L_i)$  is generally not affine under  $P_{i+1}$ .*

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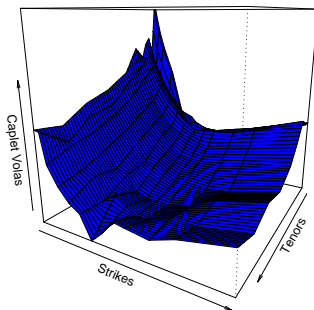
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# Caplets

The price of  $j$ -th caplet at time zero is given by

$$C_j(K) = \delta_j B_{j+1}(0) E_{P_{j+1}} [(L_j(T_j) - K)^+]$$



# Pricing Caplets under $P_{i+1}$

In terms of **log-forward moneyness**  $v = \ln(K/L_j(0))$

$$C_j(v) := \delta_j B_{j+1}(0) L_j(0) E_{P_{j+1}} [(e^{X_j(t)} - e^v)^+],$$

with  $X_j(t) = \log(L_j(t)) - \log(L_j(0))$ .

Define

$$\mathcal{O}_j(v) = E_{P_{j+1}} [(e^{X_j(t)} - e^v)^+] - (1 - e^v)^+,$$

then

$$F\{\mathcal{O}_j\}(z) := \int_{\mathbb{R}} \mathcal{O}_j(v) e^{ivz} dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)}.$$

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# Characteristic Function of $L_M$ under $P_{M+1}$

Since

$$d \ln(L_M) = -\frac{1}{2} |\Gamma_i|^2 dt + \Gamma_i^\top d\mathcal{W}^{(M+1)}(t) + d \left[ \sum_{l=1}^{N_t} \phi_i(s_l, u_l) \right]$$

and  $N_t$ ,  $\mathcal{W}^{(M+1)}$  and  $u_l$  are mutually independent

$$\Phi_{P_{M+1}}(z; T) = \Phi_{P_{M+1}}^C(z; T) \Phi_{P_{M+1}}^J(z; T),$$

where  $\Phi_{P_{M+1}}^C(z; T)$  ( $\Phi_{P_{M+1}}^J(z; T)$ ) is the c.f. of continuous (jump) part.

# Specification Analysis: Continuous Part

For some predictable vector volatility process  $(v_1(t), \dots, v_d(t))$  define

$$\Gamma_i = \begin{pmatrix} \sqrt{1 - r_{SV}^2 \gamma_{i1}} \\ \sqrt{1 - r_{SV}^2 \gamma_{i2}} \\ \vdots \\ \sqrt{1 - r_{SV}^2 \gamma_{id}} \\ r_{SV} \beta_{i1} \sqrt{v_1(t)} \\ \vdots \\ r_{SV} \beta_{id} \sqrt{v_d(t)} \end{pmatrix}, \quad \mathcal{W}^{(M+1)} = \begin{pmatrix} W_1^{(M+1)} \\ W_2^{(M+1)} \\ \vdots \\ W_d^{(M+1)} \\ \overline{W}_1^{(M+1)} \\ \vdots \\ \overline{W}_d^{(M+1)} \end{pmatrix}$$

with mutually independent  $d$ -dimensional Brownian motions  $W^{(M+1)}$  and  $\overline{W}^{(M+1)}$ .

# Specification Analysis: Continuous Part

Let

$$\gamma_i(t) = c_i g(T_i - t) \mathbf{e}_i, \quad \mathbf{e}_i \in \mathbb{R}^d,$$

where

- $c_i > 0$  are loading factors
- $g_i(\cdot)$  is a scalar volatility function
- $\mathbf{e}_i$  are unit vectors coming from the decomposition of the correlation matrix  $\zeta$

$$\zeta_{ij} = \mathbf{e}_i^\top \mathbf{e}_j, \quad 1 \leq i, j \leq M,$$

be a deterministic volatility structure of the input **Libor market model** calibrated to ATM caps and ATM swaptions.

# Specification Analysis: Continuous Part

Define a new **time independent** volatility structure via

$$\beta_i^\top \beta_j = \frac{1}{\min(i,j)} \sum_{k=1}^{\min(i,j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^\top(t) \gamma_j(t) dt.$$

## Remark

The covariance of the process  $\xi_i(t) := \int_0^t \Gamma_i^\top(t) d\mathcal{W}^{(M+1)}$  satisfies

$$\text{cov}(\xi_i(t), \xi_j(t)) \approx \int_0^t \gamma_i^\top(t) \gamma_j(t) ds$$

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# Specification Analysis: Continuous Part

Two possible specifications for the volatility process  $v$

- **Stochastic Volatility Heston Model**

$$dv_k = \kappa_k(1 - v_k)dt + \sigma_k \varrho_k \sqrt{v_k} d\bar{W}_k^{(M+1)} + \sigma_k \sqrt{(1 - \varrho_k^2)} \sqrt{v_k} dV_k^{(M+1)},$$

- **Stochastic Volatility BN Model**

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# Specification Analysis

It holds

$$\Phi_{P_{M+1}}^C(z; T) = \Phi_{D, P_{M+1}}^C(z; T) \Phi_{SV, P_{M+1}}^C(z; T),$$

where

$$\Phi_{D, P_{M+1}}^C(z; T) = \exp\left(-\frac{1}{2}\theta_M^2(T)(z^2 + iz)\right), \quad \theta_M^2(T) = \int_0^T |\gamma_M|^2 dt$$

and

$$\Phi_{SV, P_{M+1}}^C(z; T) = \exp(A_M(z; T) + B_M(z; T))$$

# Specification Analysis

In particular

$$A_M(z; T) = \frac{\kappa_M}{\sigma_M^2} \left\{ (a_M + d_M)T - 2 \ln \left[ \frac{1 - g_M e^{d_M T}}{1 - g_M} \right] \right\}$$

$$B_M(z; T) = \frac{(a_M + d_M)(1 - e^{d_M T})}{\sigma_M^2(1 - g_M e^{d_M T})}$$

and

$$a_M = \kappa_M - i \rho_M \omega_M z$$

$$d_M = \sqrt{a_M^2 + \omega_M^2(z^2 + iz)}$$

$$g_M = \frac{a_M + d_M}{a_M - d_M}, \quad \omega_M = r_{SV} \beta_{MM} \sigma_M$$

# Specification Analysis

As can be easily seen

$$\lim_{z \rightarrow \infty} \frac{A_M(z; T)}{z} = -\alpha_M \omega_M \left( i \varrho_M + \sqrt{1 - \varrho_M^2} \right) T$$

and

$$\lim_{z \rightarrow \infty} \frac{B_M(z; T)}{z} = -\frac{\sqrt{1 - \varrho_M^2} + i \varrho_M}{\sigma_M}$$

with

$$\alpha_M := \frac{\kappa_M}{\sigma_M^2}$$

# Specification Analysis

Let us take  $\phi_i(u, t) = u^\top \beta_i$ , then the characteristic function of the jump part is given by

$$\Phi_{P_{M+1}}^J(z; T) = \exp \left( \lambda T \int_{\mathbb{R}} (e^{izv} - 1) \mu_M(v) dv \right),$$

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## Observation

*Due to the Riemann-Lebesgue theorem*

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# Specification Analysis: Asymptotic Properties

Computing sequentially

$$\mathcal{L}_2 := \lim_{z \rightarrow \infty} \log(\Phi_{P_{M+1}}(z; T))/z^2,$$

$$\mathcal{L}_1 := \lim_{z \rightarrow \infty} [\log(\Phi_{P_{M+1}}(z; T))/z - (z + i)\mathcal{L}_2],$$

$$\mathcal{L}_0 := \lim_{z \rightarrow \infty} [\log(\Phi_{P_{M+1}}(z; T)) - (z^2 + iz)\mathcal{L}_2 - z\mathcal{L}_1],$$

we get

$$\mathcal{L}_0 = -\lambda, \quad \mathcal{L}_2 = -\frac{1}{2}\theta_M^2(T)$$

and

$$\operatorname{Re} \mathcal{L}_1 = -\frac{\sqrt{1 - \varrho_M^2}}{\sigma_M} - \alpha_M \omega_M \sqrt{1 - \varrho_M^2} T, \quad \operatorname{Im} \mathcal{L}_1 = -\frac{\varrho_M}{\sigma_M} - \alpha_M \omega_M \varrho_M T$$

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# Parameters Estimation: Linearization

## Observation

*From the knowledge of  $\mathcal{L}_1(T)$  for two different  $T$  one can reconstruct all parameters of the SV process*

## Theorem

$$\begin{aligned}\Psi_{P_{M+1}}(z; T) &:= \log(\Phi_{P_{M+1}}(z; T)) \\ &= \mathcal{L}_2(z^2 + iz) + \mathcal{L}_1 z + \mathcal{L}_0 + R_0 + R_1(z),\end{aligned}$$

where  $R_0 = R_0(\alpha_M, \kappa_M, \varrho_M, \omega_M)$  is a constant not depending on  $\lambda$  and

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$$R_1(z) \rightarrow 0, \quad |z| \rightarrow \infty.$$

# Parameters Estimation: Projection Estimators

We find estimates for  $\mathcal{L}_2$ ,  $\mathcal{L}_1$  and  $\mathcal{L}_0$  in the form of weighted averages

$$\begin{aligned}\widehat{\mathcal{L}}_{2,U} &:= \int \operatorname{Re}(\tilde{\Psi}_{P_{M+1}}(u)) w_2^U(u) du, \\ \widehat{\mathcal{L}}_{1,U} &:= \int \operatorname{Im}(\tilde{\Psi}_{P_{M+1}}(u)) w_1^U(u) du - i\widehat{\mathcal{L}}_{2,U}, \\ \widehat{\mathcal{L}}_{0,U} &:= \int \operatorname{Re}(\tilde{\Psi}_{P_{M+1}}(u)) w_0^U(u) du - \widehat{R}_0\end{aligned}$$

with

$$\tilde{\Psi}_{P_{M+1}}(u) := \ln \left( 1 - u(u+i)F\{\tilde{\mathcal{O}}_M\}(u+i) \right).$$

# Parameters Estimation

The weights are given by

$$w_2^U = U^{-3} w_2(u/U), \quad w_1^U = U^{-2} w_1(u/U), \quad w_0^U = U^{-1} w_0(u/U),$$

where

$$\int_{-1}^1 w_2(u) du = 0, \quad \int_{-1}^1 u w_2(u) du = 0, \quad \int_{-1}^1 u^2 w_2(u) du = 1,$$

$$\int_{-1}^1 w_1(u) du = 0, \quad \int_{-1}^1 u w_1(u) du = 1,$$

$$\int_{-1}^1 w_0(u) du = 1, \quad \int_{-1}^1 u w_0(u) du = 0, \quad \int_{-1}^1 u^2 w_0(u) du = 0.$$

# Parameters Estimation: Jump distribution

Define

$$F\{\hat{\mu}_M\}(z) = \tilde{\Psi}_{P_{M+1}}(z; T) - \hat{\mathcal{L}}_2(z^2 + iz) - \hat{\mathcal{L}}_1 z - \hat{\mathcal{L}}_0 - \hat{R}_0$$

or equivalently

$$\hat{\mu}_M := F^{-1} \left[ \left( \tilde{\Psi}_{P_{M+1}}(\cdot; T) - \hat{\mathcal{L}}_2(\cdot^2 + i\cdot) - \hat{\mathcal{L}}_1 \cdot - \hat{\mathcal{L}}_0 - \hat{R}_0 \right) \mathbf{1}_{[-U, U]}(\cdot) \right]$$

## Remark

*Due to lack of data and numerical errors  $\hat{\mu}_M$  may not be a density and needs to be corrected.*

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# Parameters Estimation: Further optimization

Upon finding

$$\left(\widehat{\mathcal{L}}_{0,U}, \widehat{\mathcal{L}}_{1,U}, \widehat{\mathcal{L}}_{2,U}\right) \longrightarrow \widehat{\mathcal{T}} := \left(\widehat{\sigma}_M, \widehat{\varrho}_M, \widehat{\kappa}_M, \widehat{\lambda}\right)$$

we may

- consider  $\widehat{\mathcal{T}}$  as a final set of parameters or
- consider nonlinear least-squares

$$\mathcal{J}(\mathcal{T}) = \sum_{i=1}^N w_i |C_M^{\mathcal{T}}(K_i) - C_M(K_i)|^2$$

and minimize  $\mathcal{J}(\mathcal{T})$  over the parametric set  $\mathcal{S} \subset \mathbb{R}^4$  taking as initial value  $\widehat{\mathcal{T}}$ .

# Approximative dynamics of $L_i$ under $P_{i+1}$

It holds approximately

$$\frac{dL_i}{L_i} \approx \Gamma_i^\top dW^{(i+1)} + \int_E e^{u^\top \beta_j} (\mu - \tilde{\nu}^{(i+1)})(dt, du),$$

where  $dW^{(i+1)}$  is a standard Brownian motion under  $P_{i+1}$  and

$$\tilde{\nu}^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[ \prod_{j=i+1}^M \left( 1 + \frac{\delta_j L_{j-}(0) e^{u^\top \beta_j}}{1 + \delta_j L_{j-}(0)} \right) \right].$$

## Approximative dynamics of $v_k$ under $P_{i+1}$

By freezing the Libors at their initial values we obtain an approximative  $v_k$  dynamics

$$dv_k \approx \kappa_k^{(i+1)} \left( \theta_k^{(i+1)} - v_k \right) dt + \sigma_k \sqrt{v_k} \left( \varrho_k d\widetilde{W}_k^{(i+1)} + \sqrt{1 - \varrho_k^2} d\overline{W}_k^{(i+1)} \right)$$

with reversion speed parameter

$$\kappa_k^{(i+1)} := \left( \kappa_k - r_{SV} \sigma_k \varrho_k \sum_{j=i+1}^M \frac{\delta_j L_j(0)}{1 + \delta_j L_j(0)} \beta_{jk} \right),$$

and mean reversion level

$$\theta_k^{(i+1)} := \frac{\kappa_k}{\kappa_k^{(i+1)}}.$$

## Pricing Caplets under $P_{M+1}$

The price of  $j$ -th caplet at time zero can be alternatively written as

$$C_j(K) = \delta_j B_{M+1}(0) E_{P_{M+1}} \left[ \frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} (L_j(T_j) - K)^+ \right]$$

Note that

$$\begin{aligned} \frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} &= \prod_{k=j+1}^M (1 + \delta_k L_k(T_j)) \\ &= \prod_{k=j+1}^M (1 + \delta_k) E_{\xi} \exp \left( \sum_{k=j+1}^M \xi_k \ln(L_k(T_j)) \right), \end{aligned}$$

where  $\{\xi_k\}_{k=j+1}^M$  are independent random variables and each  $\xi_k$  takes two values 0 and 1 with probabilities  $1/(1 + \delta_k)$  and  $\delta_k/(1 + \delta_k)$ .

# Pricing Caplets under $P_{M+1}$

Thus,

$$F\{\mathcal{O}_j\}(z) = \frac{1 - E_\xi \Phi_{M+1}(z - i, \xi_{j+1}, \dots, \xi_M)}{z(z - i)},$$

where  $\Phi_{M+1}(z_j, z_{j+1}, \dots, z_M)$  is the joint characteristic function of  $(\ln(L_j(T_j)), \dots, \ln(L_M(T_j)))$  under  $P_{M+1}$ .

## Remark

*Instead of terminal measure  $P_{M+1}$  we could consider  $P_{I+1}$  with  $1 < I < M + 1$ .*

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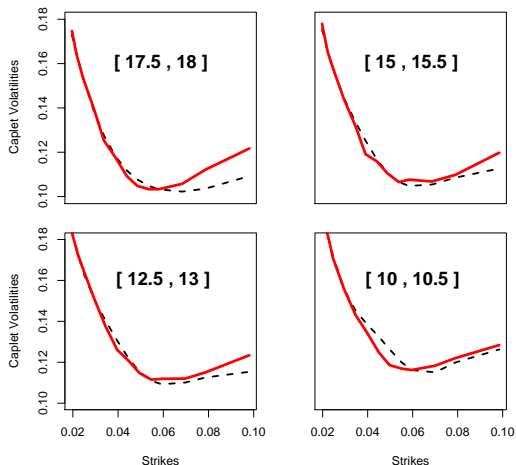
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# Calibration results for 14.08.2007

## Caplet volas for different caplet periods





Belomestny, D. and Spokoiny, V.

Spatial aggregation of local likelihood estimates with applications to classification,

*Annals of Statistics*, 2007, **35**(5), 2287–2311.



Belomestny, D. and Reiß, M.

*Spectral calibration of exponential Lévy models*,

*Finance and Stochastics*, 2006, **10**(4), 449–474.



Belomestny, D. and Schoenmakers, J.

A jump-diffusion Libor model and its robust calibration,

SFB649 Discussion Paper, 2006, **037**.