Stable calibration methods for equity models of local Lévy type

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Joint work with S. Kindermann, H. Albrecher & H.W. Engl

Outline

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The usual way of pricing of derivatives

1. Model for the risk-neutral dynamics of the underlying asset (e.g. via stochastic differential equation)
2. Free (unknown) parameters in the model (e.g. volatility in BS-model)
3. Identification of parameters by observed option prices (Inverse Problem).
4. Calibrated model used for pricing illiquid derivatives (e.g. via MC-simulation, solving P(I)DE'S,...)

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Problems:

- √ Data not exact (e.g. bid-ask-spreads)
- √ Discrete data sets (not complete option-price surface known)
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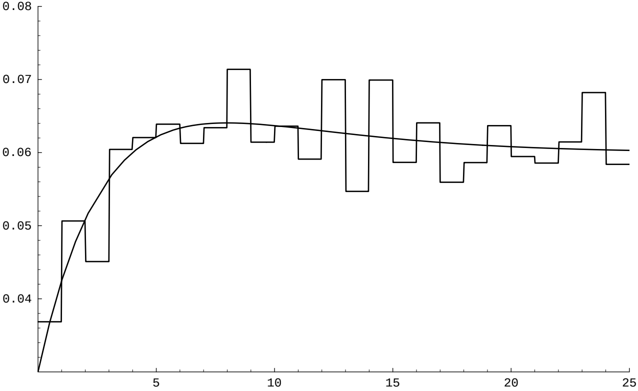
What can happen with ill-posed problems?

Calculate (deterministic) short rate from given bond-prices
(piecewise constant, noiselevel $< 1\%$)
yearly observations used



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Calculate (deterministic) short rate from given bond-prices
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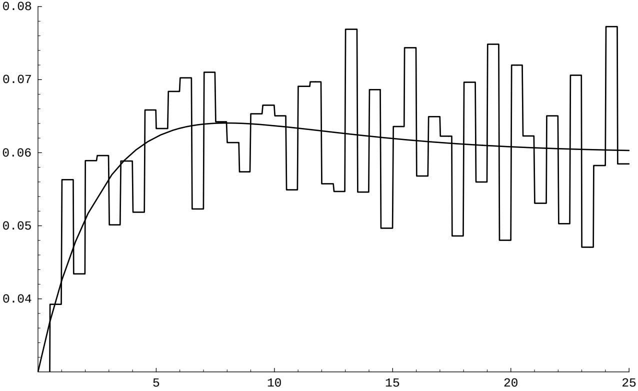

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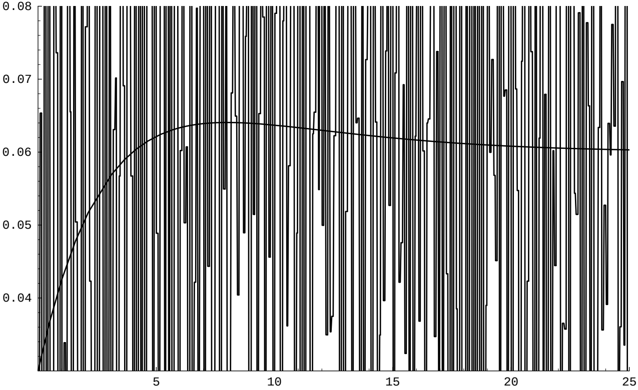

How to circumvent the problems?

Facts:

- √ dangerous to fit in a naïve way
- √ more (noisy) data $\nRightarrow$ more accurate results

Possible correction: Regularization

How to circumvent the problems?

Facts:

- √ dangerous to fit in a naïve way
- √ more (noisy) data $\not\Rightarrow$ more accurate results

Possible correction: Regularization

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What has been done so far

Application of regularization in computational finance:

- √ *S. Crepey*: Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization. (2003)
- √ *H. Egger & H. W. Engl*: Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates. (2005)
- √ *R. Cont & P. Tankov*: Nonparametric calibration of jump-diffusion option pricing models. (2004)
- √ *R. Cont & P. Tankov*: Recovering exponential Lévy models from option prices: regularization of an ill-posed inverse problem. (2006)

The well known local volatility model

Dupire (1994), Derman & Kani (1994):

$$dS_t = (r - \eta)S_t dt + \sigma(S_t, t)S_t dW(t),$$

r riskless interest rate, η dividend yield

$\surd$ capable of fitting marginal distributions of any Itô process (Gyöngy, 1986)

$\Rightarrow$ Parabolic PDE for call price

$$C_T + \eta C + (r - \eta)KC_K - \frac{1}{2}\sigma^2(K, T)K^2 C_{KK} = 0$$

$$C(0, K) = (S_0 - K)^+$$

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Drawbacks of the local volatility model

- √ often very steep volatility structure
- √ Problems when pricing path-dependent options
- √ No jumps possible

Possible improvement: Lévy models

Advantages:

- √ jump-risk included
- √ fat tails
- √ skewed log-returns via asymmetric Lévy measure

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In analogy to local volatility-models: local Lévy model (Carr et al. 2004)

- Parameter of Brownian motion & Lévy measure dependent on S_t and t:

$$dS_t = (r-\eta)S_{t-}dt + \sigma_0(S_{t-},t)S_{t-}dW(t) \quad (1)$$

$$\int_{-\infty}^{\infty}(e^x-1)\left(m_{(S_{s-},s)}(dx,du) - \nu_{(S_{s-},s)}(dx,du)\right) \quad (2)$$

W . . . Brownian motion

m . . . integer valued random measure independent of W

ν . . . its compensator

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Special state-dependence of the Lévy measure

- √ Local volatility models: Brownian motion with variable speed (depending on local volatility function)
- √ Carr et al. variable speed of jump-part: local speed function

$$\nu_{(S_{t-},t)}(dx, dt) = a_0(S_{t-}, t)\nu(dx, dt)$$

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Solvability of the SDE

other representation:

$$S_t = S_0 e^{(r-\eta)t} e^{X_t},$$

where X_t is a semimartingal with characteristics

$$(-\sigma^2(\xi_{t-}, t)/2, \sigma(\xi_{t-}, t), a(\xi_{t-}, t)dt \times \nu),$$

where: $\xi_t = \ln(S_0) + (r - \eta)t + X_t$, $\sigma(x, t) = \sigma_0(e^x, t)$ and $a(x, t) = a_0(e^x, t)$.

Note: speed function does not affect the jump-size distribution

Existence results: e.g. Ethier & Kurtz (1986), Gihman & Skorohod (1979) under different conditions

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Option price for local Lévy model I

Carr et al.: with Tanaka-Meyer formula partial integro-differential equation (PIDE) for callprice:

$$
\begin{aligned}
C_T &= \eta C + (r-\eta) K C_K + \frac{\sigma^2(K,T)}{2} K^2 C_{KK} \\
&\quad + \int_0^\infty Y C_{YY}(Y,T) a_0(Y,T) \psi_e \left(\log \left(\frac{K}{Y} \right) \right) dY
\end{aligned}
$$

in the weak sense

Option price for local Lévy model II

Structure of Equation

$$\begin{aligned} C_T &= -\eta\, C - (r-\eta) K\, C_K + \frac{\sigma_0^2(K,T)}{2} K^2\, C_{KK} \\ &\quad + \int_0^\infty Y\, C_{YY}(Y,T) a_0(Y,T) \psi\left(\log\left(\frac{K}{Y}\right)\right) dY \end{aligned}$$

Black Scholes Part + Integral Operator with kernel ψ

- ✓ σ_0 . . . local volatility
- ✓ a_0 . . . local speed function
- ✓ ψ . . . double exponential tail of Lévy measure

$$\psi(z) = \begin{cases} \int_{-\infty}^{z} (e^z - e^x)\, \nu(dx) & \text{for } z < 0 \\ \int_z^\infty (e^x - e^z)\, \nu(dx) & \text{for } z > 0. \end{cases}$$

What is the task?

$\Rightarrow$ Parameter identification in a PIDE using partial knowledge of solution

For Black-Scholes:

$$\sigma^2(K,T) = 2\frac{C_T - \eta C - (r-\eta)KC_K}{K^2 C_{KK}}$$

Problem:

Differentiation of Data $C(K,T) \Rightarrow$ ill-posed (2 times differentiation)

- ✓ Not stable for noisy data
- ✓ Small perturbation in data can lead to large deviation in parameter

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Finding the local speed function

Assume σ & ψ fixed:

$$a(K,T) = \frac{b(\log(K),T)}{K^2 C_{KK}}$$

$$\int_{-\infty}^{\infty} b(y,T)\psi(k-y)\,dk = C_T(e^k) + \eta C(e^k,T) + (r-\eta)e^k C_K(e^k,T) - \frac{\sigma^2(e^k,T)}{2} e^{2k} C_{KK}(e^k,T)$$

Observation: in contrast to Dupire model + a convolution equation (convolution also ill-posed)

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Abstract formulation of the problem

Identification problem as operator equation:

$$F(\theta) = y$$

$$F : \theta \to C(K, T)$$

F ... parameter to solution operator
y ... data = (observed option prices)
θ ... (σ, a, ν)

Problem ▶ F does not have continuous inverse.

$\Rightarrow F(\theta) = y$ cannot be solved stably!

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No direct solution - what else?

Wishlist:

- √ Stability: Computed solution should depend continuously on data
- √ Approximation: Computed solution should be close to solution of Equation

Two demands are in opposition to each other

Regularization: compromise

Nonlinear Problem ⇒ Nonlinear Tikhonov Regularization

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Tikhonov regularization

Find approximate solution by minimizing

$$J(\theta) := \|F(\theta) - y_\delta\|^2 + \alpha\|\theta - \theta^*\|^2$$

y_δ . . . noisy data
δ . . . noise level: $\|y_\delta - y\| = \delta$
θ^* . . . initial guess
$\|.\|$ . . . suitable norms
α . . . regularization parameter

Tikhonov Regularization

- ✓ for local volatility identification: e.g. Crepey, Engl & Egger
- ✓ Jump-diffusion Lévy Model: Cont & Tankov

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Convergence theory for Tikhonov regularization

Nonlinear Theory: Engl, Kunisch & Neubauer (1989), Engl, Hanke & Neubauer (1996)

Results:

If graph of F is weakly sequentially closed

- ✓ Minimizer of Tikhonov functional exists
- ✓ Minimizer depends continuously on data for $\alpha > 0$
- ✓ For noise-free data: Minimizer converges to true parameter as $\alpha \to 0$
- ✓ Noisy data: if $\frac{\delta^2}{\alpha(\delta)} \to 0$, regularized solution converges to true solution as noise level $\delta \to 0$.

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Identification of the local speed function

Assume for some reasons $\sigma > 0$, ψ fixed & T^* a finite planning horizon:

Operator equation:

$$F(a) = y,$$

where y is the given data and F defined via PIDE

For applicability of Tikhonov regularization:

- √ Well-posedness of forward problem (cf. Matache et al. 2004, 2005)
- √ Continuity of F

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Reparametrization of the call price

$$c^{(\theta)}(k,T) = e^{\eta T}\,\mathrm{C}(\mathrm{e}^k,T) - g(x),$$

where $g(k) = \left(S - \mathrm{e}^k\right)^+$

Introducing

$$\mathcal{I}_\psi u \;:=\; \psi * u = \int_{-\infty}^{\infty} \psi(k-y)u(y)\,dy$$

$$\mathcal{L}_\theta u \;:=\; \left(r - \eta + \frac{\sigma^2(k,T)}{2}\right)u_k - \frac{\sigma^2(k,T)}{2}u_{kk} - \mathcal{I}_\psi\big(a(.,T)(u_{kk} - u_k)\big)$$

$$c_T^{(\theta)} + \mathcal{L}_\theta c^{(\theta)}(.,T) = -\mathcal{L}_\theta g,$$

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Existence and Uniqueness of PIDE solution

Assumptions:

$$\sigma(k,T) \geq c_0 > 0 \in L^\infty([0,T^*], W^{1,\infty}(\mathbb{R}))$$
$$\sigma_k(k,T) \in L^\infty([0,T^*], L^2(\mathbb{R})).$$

(Positivity and Smoothness of volatility)

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$$a_k(k,T) \in L^\infty([0,T^*], L^2(\mathbb{R})).$$

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$$\mathbb{E}\left[S_t \ln S_t\right] < \infty, \quad 0 \leq t \leq T^*$$

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$\mathbb{E}\left[S_t \ln S_t\right] < \infty \Leftrightarrow \psi_e$ integrable

Existence and Uniqueness of PIDE solution

Assumptions:

$$
\begin{aligned}
\sigma(k,T) &\geq c_0 > 0 \in L^\infty([0,T^*],W^{1,\infty}(\mathbb{R})) \\
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Well-posedness of forward problem

Existence and Uniqueness by Gårding inequality:

Black Scholes Part:

$$c_T^{(\theta)} - (r - \eta - \frac{\sigma^2(k,t)}{2})c_k^{(\theta)} + \frac{\sigma^2(k,T)}{2}c_{kk}^{(\theta)}$$

$\Rightarrow$ standard parabolic theory

Integral part

integral term bounded by splitting up and using Sobolev embedding theorems

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The good news

Under assumptions on a, σ, ν:

- ✓ $c^{(\theta)}(K,T)$ depends continuously on a (in appropriate norms)
- ✓ $c^{(\theta)}(K,T)$ is Frechet-differentiable with respect to a
- ✓ Frechet-derivative is Lipschitz continuous

$\Rightarrow$ Theory of Nonlinear Tikhonov Regularization is applicable

For our problem:

- ✓ Existence of a Minimizer
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For ill-posed problems convergence of regularized solution can be arbitrarily slow. (Schock 1985).

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$a^\dagger$ ... true solution

Source condition

$$a^\dagger - a^* \in R(F'^*(a^\dagger))$$

F'^*... adjoint of the Frechet-derivative
a^* ... initial guess

- ✓ Range of F'^* is smooth
- ✓ Source condition ⇒ initial guess sufficiently good

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Convergence Rates for Nonlinear Tikhonov Regularization

- √ If F is differentiable and F' is Lipschitz
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- ✓ Smoother range of $F'^*(a^\dagger) \Rightarrow$ sharper the source condition.
- ✓ Smoother $F'^* \Rightarrow$ Problem more difficult

$$F'^*(a^\dagger) \sim K^2 C_{KK} \mathcal{I}(v)$$

$\mathcal{I}$ is integral operator with kernel ψ
v is solution to adjoint PIDE
C option price for exact speed function $a^\dagger$

degree of ill-posedness: Ill-posedness of two problems can be compared: Higher smoothing $F'^* \Leftrightarrow$ a more ill-posed problem.

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Interpretation of source condition II

Consequences from source condition

Multiplication part:

$$F'^{*}(a^{\dagger}) \sim K^2 C_{KK} I^{*} v$$

Wherever $C_{KK}(K,T) = 0$ the exact solution has to be known!.

- ✓ Interpretation: $C_{KK}(K,T) \sim$ density of S_T
- ✓ If density is 0 at some K, T then speed function $a_0(K,T)$ has no influence on solution.
- ✓ $\Rightarrow$Not uniquely identifyable there.

For positive volatility $\sigma_0 > 0$ density positive $(C_{KK} > 0)$
Not necessarily so in more general Lévy model

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Comparison with local volatility model

source conditions of local Lévy and local volatility

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Identification for Local Lévy problem more ill-posed than local volatility identification (smoothing integral operator)

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Ill-Posedness

For typical case ψ: jump at 0 (or not defined)

$$\int_0^\infty (e^x - 1)\nu(dx) \neq \int_{-\infty}^0 (1 - e^x)\nu(dx)$$

$\Rightarrow \mathcal{I}$ like a smoothing operator of order 1.

Difference of ill-posedness between Local Lévy and Local Volatility

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$\sim$ one additional differentiation

Compare: data differentiation formula

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Numerical Computation

Task: Find the global minimum of Tikhonov regularization

$$a_{\alpha.\delta} = \text{argmin}(\|F(a) - y_\delta\|^2 + \alpha\|a - a^*\|_s^2),$$

where $\|.\|_s$ the tensor product norm $H^1[0, K_0] \otimes H^1[0, T]$

- √ Discretization of forward operator and parameters
- √ Discretization of PIDE (finite element + Crank Nicholson)
- √ Optimization routine by Gauss-Newton method (locally convergent)
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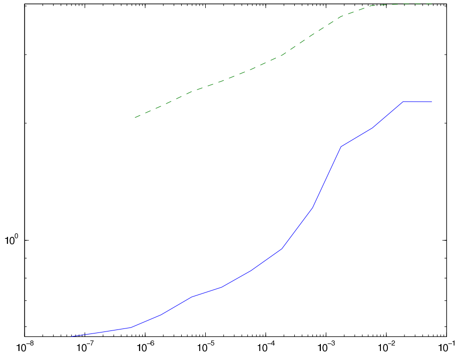

Rates depend on smoothness of $a^\dagger$

Numerical Computation Synthetic data

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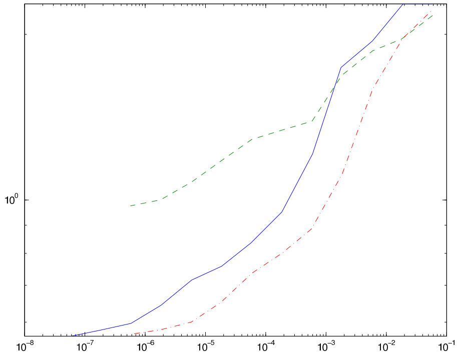
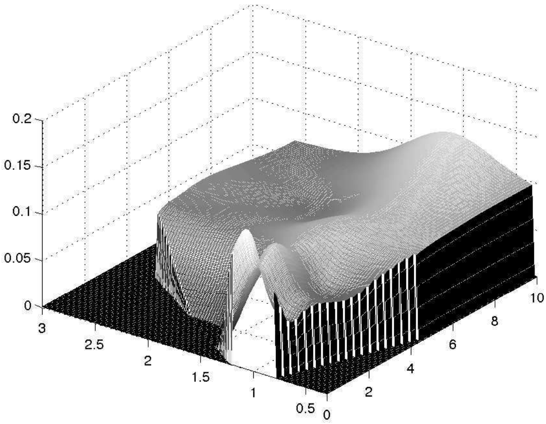


Rates depend on smoothness of kernel ψ_e

Numerical Computation Real data

Computed solution of local speed function: For Option price data from Andersen & Andreasen



Outline

Identification of σ

only ν fixed
$\Rightarrow$ identify local volatility $\sigma > 0$ & local speed function a

Problem:

$$F(\sigma, a) = y$$

with same techniques, i.e. PIDE-methods & Tikhonov regularization, feasible

Frechet derivative:

- √ exists $\Rightarrow$ computationally tractable
- √ locally Lipschitz continuous $\Rightarrow$ convergence rates (if source condition met)

Application $\Rightarrow$ local time for jump diffusion

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- ✓ norm on density
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Conclusions

- √ Nonlinear Tikhonov Regularization can be used as a stable and robust method for identifying parameters in financial stochastic differential equations.
- √ Identifying parameters in local Lévy model is more ill-posed than in local volatility model
- √ Theory of Tikhonov Regularization allows generalisation to other related parameter identification problems

Thank you for your attention!