Reducing the Risk of Optimal Portfolio Policies

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Joint work with

- Ulrich Haussmann (Vancouver),
- Abdelali Gabih (Halle), Ralf Wunderlich (Zwickau),
- Markus Hahn, Wolfgang Putschögl (RICAM) in the FWF Project

Computing Optimal Portfolio Policies
under Partial Information.

In addition joint work on parameter estimation in Markov switching models with Robert J. Elliott (Calgary), Sylvia Frühwirth-Schnatter (Linz) and Vikram Krishnamurthy (Vancouver).

The Merton strategy

In the Black-Scholes model stock prices satisfy

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \quad \textbf{i.e.} \quad S_t = S_0\, \mathrm{e}^{\mu\, t}\, \mathrm{e}^{\sigma\, W_t - \sigma^2/2\, t}\ .$$

The optimal strategy for maximizing expected utility of terminal wealth,

$$\textbf{maximize} \quad \mathrm{E}[U(X_T)],$$

is given by the constant risky fraction

$$\hat{f}_t = \frac{1}{RRA}\,\frac{\mu - r}{\sigma^2} \quad t \in [0, T] \quad \textbf{(Merton)}$$

for certain utility functions with constant relative risk aversion RRA**.**

If $0 < \hat{f} < 1$ **this implies:**

Buy in a bearish market, sell in a bullish market

A hidden Markov model for the stock returns

We consider one money market with interest rate $r = 0$
and one stock with prices S and returns R,

$$dS_t = S_t \, dR_t \ , \quad dR_t = \mu_t \, dt + \sigma \, dW_t \ , \quad t \in [0, T] \ ,$$

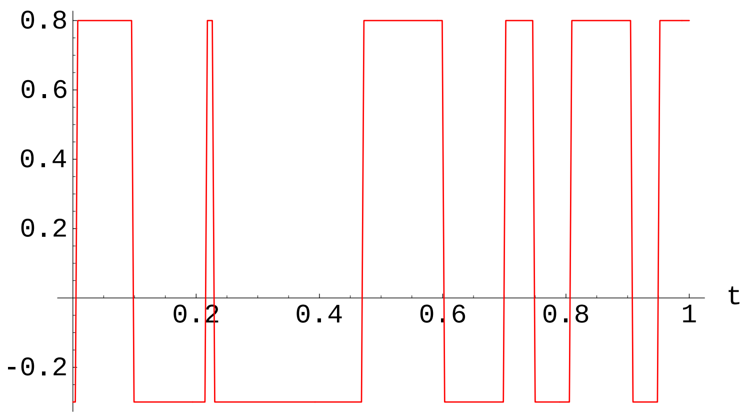
where W is a Brownian motion, W and μ are independent,
μ is a continuous time Markov chain given by

$$\mu_t = b^\top Y_t \ , \quad b = (b_1, \ldots, b_d) \quad \text{state vector,}$$

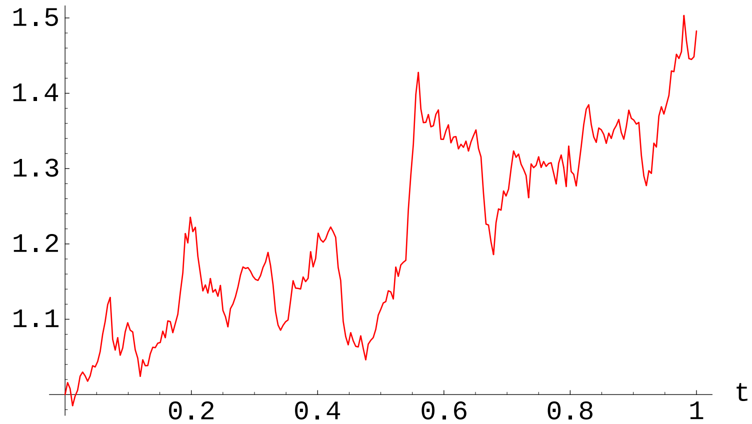
and Y is a continuous time Markov chain with states $e_1, \ldots, e_d$ and rate matrix Q. So

$$\mu_t = b_k \quad \text{if and only if} \quad Y_t = e_k, \quad k = 1, \ldots, d.$$

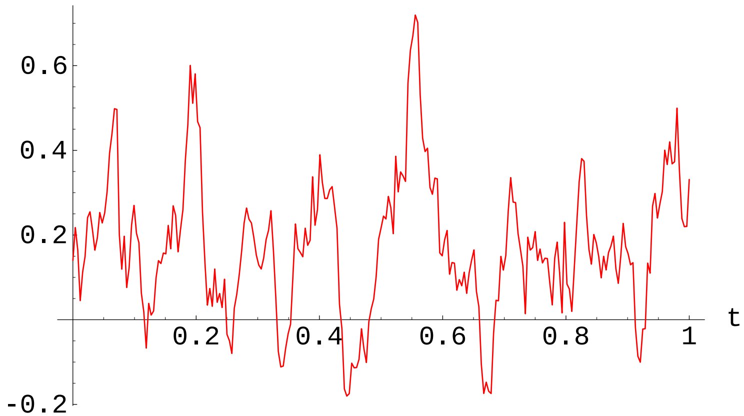
Simulation of one path



Drift process μ



Stock prices S



Filter $\mathrm{E}[\mu_t \,|\, \mathcal{F}_t^R]$

Parameters: $\sigma = 0.25$,

$$b = \begin{pmatrix} 0.74 \\ -0.36 \end{pmatrix}, \quad Q = \begin{pmatrix} -15 & 15 \\ 10 & -10 \end{pmatrix}.$$

For daily trading $\Delta t = 1/252$

$$\begin{aligned} b\Delta t &\approx (0.0029, -0.0014)^\top, \\ \sigma\sqrt{\Delta t} &\approx 0.0157. \end{aligned}$$

Optimizing the terminal wealth under partial information

- **trading strategy** $\pi = (\pi_t)_{t\in[0,T]}$ **:** $\mathcal{F}^R$**-adapted;** π_t **money in stock,**
- **wealth process** $X^\pi = (X_t^\pi)_{t\in[0,T]}$**,**

$$dX_t^\pi = \pi_t dR_t = \pi_t(\mu_t\, dt + \sigma\, dW_t)\ , \qquad X_0^\pi = x_0\ ,$$

- **utility function** $U : (0,\infty) \to I\!R$ **: Strictly concave, strictly increasing, twice continuously differentiable with** $U'(0+) = \infty$**,** $U'(\infty) = 0$**.**

  E.g. $U_0(x) = \log(x)$ **and power utility** $U_\alpha(x) = x^\alpha/\alpha$**,** $\alpha < 1$**,** $\alpha \neq 0$**.**

Optimization Problem: Solve

$$V(x_0) = \textbf{sup} \left\{ \mathrm{E}[U(X_T^\pi)] \mid \pi \quad \textbf{with } X^\pi \geq 0 \right\} ,$$

i.e. find $\hat\pi$ **with** $\mathrm{E}[U(\hat X_T)] = V$**,** $\hat X \geq 0$**.**

Solution for logarithmic utility

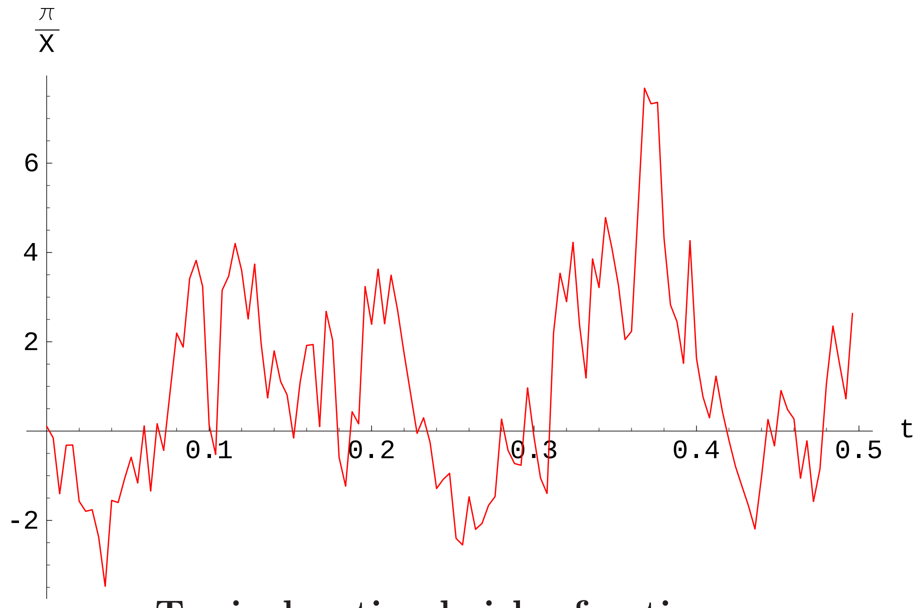
For $U(x) = \log(x)$ **we get**

$$\frac{\hat{\pi}_t}{\hat{X}_t} = \frac{b^\top \eta_t}{\sigma^2} = \frac{\mathrm{E}[\mu_t \,|\, \mathcal{F}_t^R]}{\sigma^2} \,.$$

For $\sigma = 0.25, \quad b = \begin{pmatrix} 0.74 \\ -0.36 \end{pmatrix}$

we then have

$$-5.76 < \frac{\hat{\pi}_t}{\hat{X}_t} < 11.84 \,.$$



Typical optimal risky fraction

strategy	av. X_T	m. X_T	av. $\log(X_T)$	m. $\log(X_T)$	aborted
b/h	**1.153**	**1.121**	**0.116**	**0.114**	–
Merton	**1.136**	**1.089**	$-\infty$ **(-0.016)**	**0.086**	**2**
optimal	**1.520**	**1.145**	$-\infty$ **(0.042)**	**0.135**	**12**

Application to historical prices (averages, medians of 500**, see below)**

Possible Improvements

We can try

- to improve the model or (and)
- to impose suitable constraints.

E. g.,

stay in complete market	lead to incomplete market
risk averse utility functions	...
non-constant interest rates	...
non-constant volatility	stochastic volatility
risk constraints (on the wealth)	convex constraints (on the strategy)
	Lévy noise

Determination of the optimal trading strategy

Introduce $d\tilde{P} = Z_T\, dP$ **by** $dZ_t = -Z_t\, \sigma^{-1} \mu_t\, dW_t$**, set** $\tilde{W}_t = W_t + \int_0^t \sigma^{-1} \mu_s\, ds$**. Then**

$$R_t = \int_0^t b^\top Y_s\, ds + \sigma\, W_t = \sigma\, \tilde{W}_t\ .$$

Filtering: **By Bayes' Theorem**

$$\eta_t = \mathrm{E}[Y_t \mid \mathcal{F}_t^R] = \frac{\tilde{\mathrm{E}}[Z_T^{-1} Y_t \mid \mathcal{F}_t^R]}{\tilde{\mathrm{E}}[Z_T^{-1} \mid \mathcal{F}_t^R]} = \frac{\mathcal{E}_t}{\mathbf{1}^\top \mathcal{E}_t}.$$

Then $\mathrm{E}[\mu_t \,|\, \mathcal{F}_t^R] = b^\top \mathrm{E}[Y_t \,|\, \mathcal{F}_t^R] = b^\top \eta_t$ **and** $\zeta_t = \mathrm{E}[Z_t \mid \mathcal{F}_t^R] = 1 \,/\, \mathbf{1}^\top \mathcal{E}_t$**.**

Under $\tilde{P}$ **we get** $\quad X_T^\pi = x_0 + \int_0^T \pi_t \sigma\, d\tilde{W}_t\ ,\quad$ **hence** $\tilde{\mathrm{E}}[X_T^\pi] = x_0\ .$

A general method for solving $\max_\pi \mathrm{E}[U(X_T^\pi)]$ **:**

Convex analysis $\Rightarrow$ $\hat{X}_T = I(\hat{y}\,\zeta_T)$, $\tilde{\mathrm{E}}[I(\hat{y}\,\zeta_T)] = x_0$ **with** I **inverse of** U'

Martingale-representation ($\mathcal{F}^R = \mathcal{F}^{\tilde{W}}$) $\Rightarrow$ **existence of** $\hat{\pi}$

Clark's formula $X_T^\pi = x_0 + \int_0^T \tilde{\mathrm{E}}[D_t X_T^\pi | \mathcal{F}_t^R]\, d\tilde{W}_t \Rightarrow \hat{\pi}$ **in terms of** $D_t \hat{X}_T$**.**

The optimal trading strategy

Let the m-dimensional state process $\xi = (\xi_t)_{t\in[0,T]}$ be given by

$$d\xi_t = \nu(\xi_t)\,dt + \tau(\xi_t)\,d\tilde{W}_t\ ,$$

and let $\quad \sigma_t = \sigma(\xi_t), \quad r_t = r(\xi_t), \quad t \in [0,T]\ .$

The model guarantees a complete market w.r.t. $\mathcal{F}^R = \mathcal{F}^S = \mathcal{F}^{\tilde{W}}$.

The excess return process $\tilde{R}$ becomes

$$d\tilde{R}_t = dR_t - r_t\,dt = (b - r_t\mathbf{1})^\top Y_t\,dt + \sigma_t\,dW_t = \sigma_t\,d\tilde{W}_t.$$

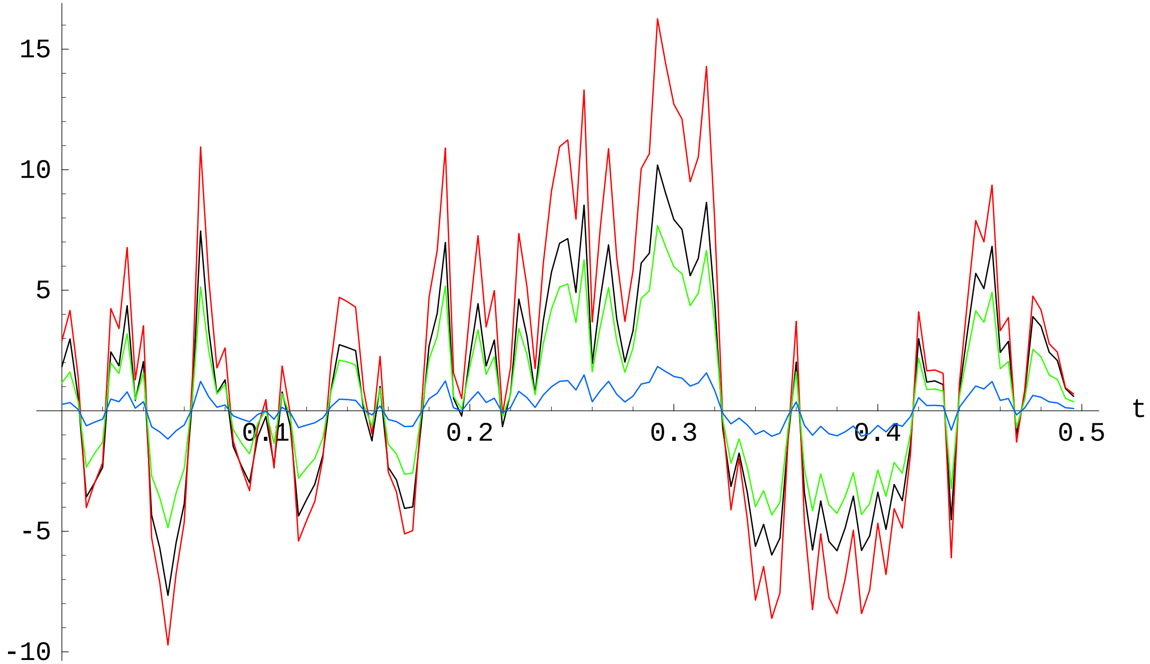
Theorem 1 *Denoting* $\beta_t = \exp(-\int_0^t r_s\,ds)$, $\vartheta_t = \sigma_t^{-1}(b - r_t\mathbf{1})$, *for* $U_\alpha = x^\alpha/\alpha$

$$\begin{aligned}\frac{\hat{\pi}_t}{\hat{X}_t} \;=\; & \frac{\sigma_t^{-1}}{(1-\alpha)\tilde{\mathrm{E}}[\beta_{t,T}\tilde{\zeta}_{t,T}^{\frac{1}{\alpha-1}}\mid\xi_t,\mathcal{E}_t]}\Big\{\vartheta_t^\top\eta_t\tilde{\mathrm{E}}[\tilde{\zeta}_{t,T}^{\frac{\alpha}{\alpha-1}}\mid\xi_t,\mathcal{E}_t]\\ & + \tilde{\mathrm{E}}[\tilde{\zeta}_{t,T}^{\frac{\alpha}{\alpha-1}}\int_t^T((D_t\mathcal{E}_s)\vartheta_s + (D_t\vartheta_s)\mathcal{E}_s)\,d\tilde{W}_s\mid\xi_t,\mathcal{E}_t]\\ & + \alpha\,\tilde{\mathrm{E}}[\beta_{t,T}\int_t^T(D_t r_s)\,ds\mid\xi_t,\mathcal{E}_t]\Big\},\end{aligned}$$

where $\beta_{t,T} = \beta_T/\beta_t$, $\zeta_{t,T} = \zeta_T/\zeta_t$, $\mathcal{E}_{t,s} = \mathcal{E}_s/\zeta_t^{-1}$, $\tilde{\zeta}_t = \beta_t\zeta_t$.

Optimal risky fractions for constant σ

For utility functions $U_0(x) = \log(x)$ **and** $U_\alpha(x) = x^\alpha/\alpha$, $\alpha < 1$, $\alpha \neq 0$:



Optimal risky fractions $\hat{\pi}/\hat{X}$ **for** $\alpha = 0.2$, $\log$, $\alpha = -0.5$, $\alpha = -5$.

Some examples of non-constant volatility models

- **Constant elasticity of variance model**:

$$dS_t = S_t(\mu_t\, dt + \sigma_0\, S_t^{\beta/2-1})dW_T$$

corresponding to

$$\sigma_t = \xi_t, \quad d\xi_t = \frac{\xi_t^3}{2}\left(\frac{\beta}{2}-2\right)\left(\frac{\beta}{2}-1\right)dt + \xi_t^2\left(\frac{\beta}{2}-1\right)d\tilde{W}_t\ .$$

- **Hobson-Rogers model**:

$$\sigma_t = \sigma_0\sqrt{1+\varepsilon\,\xi_s^2}\wedge \text{bigconstant}, \quad d\xi_t = -\left(\frac{1}{2}\sigma_t^2 + \lambda\xi_t\right)dt + \sigma_t d\tilde{W}_t\ ,$$

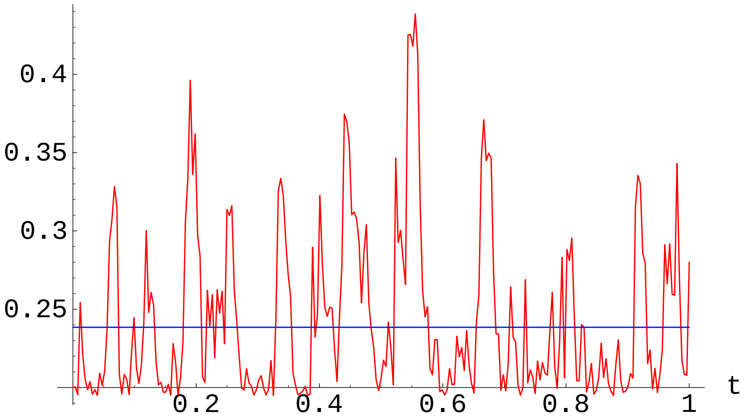
based on the first offset function

$$\xi_t = \int_0^\infty \lambda\, \mathrm{e}^{-\lambda\, u}(\log S_t - \log S_{t-u})du\ .$$

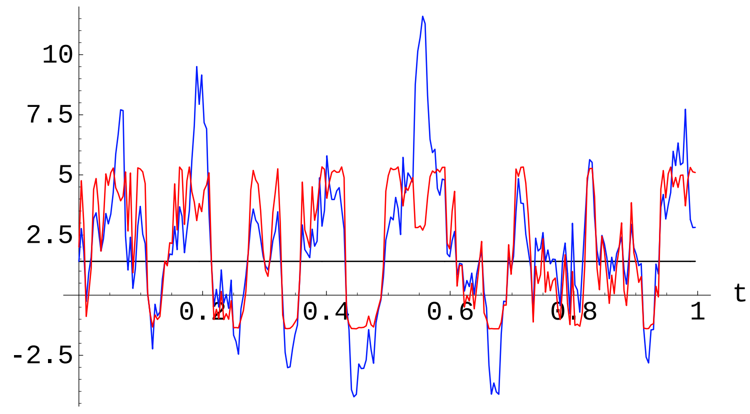
- **Filter based volatility: In the case of two states**

$$\sigma_t = \sigma(\eta_t^1) = s_0 + s_1\eta_t^1 + s_2(\eta_t^1)^2, \quad \xi_t = \eta_t\ .$$

Simulation of one path and application to market data



Volatility process σ (filter model)



Optimal risky fraction π/X

α	uses r_t,	faces r_t	0	uses $\bar{r}$,	faces r_t	0	uses $\bar{r}$,	faces r_t	-5
strategy	$\hat{X}_T$	$\log(\hat{X}_T)$	< 0	$\hat{X}_T$	$\log(\hat{X}_T)$	< 0	$\hat{X}_T$	$-\frac{1}{5}\hat{X}_T^{-5}$	< 0
stoch. σ	**1.745**	**0.122**	**1**	**1.646**	**0.106**	**1**	**1.116**	**-0.132**	**0**
const. σ	**1.603**	**0.057**	**11**	**1.529**	**-0.042**	**12**	**1.136**	**-0.633**	**0**
Merton	**1.163**	**0.006**	**2**	**1.137**	**-0.020**	**3**	**1.097**	**-0.142**	**0**
b/h	**1.153**	**0.116**	–	**1.153**	**1.121**	–	**1.153**	**-0.262**	–

Averages of optimal terminal wealth for historical prices, see below

Remarks

- Also Hobson-Rogers model, level dependent volatility models can be included. The former improves the performance, the latter not.
- For known dynamics of r and σ, strategies can be calculated based on the observed stock prices using Monte Carlo methods to compute the conditional expectations at each time step.
- The improved performance relies on avoiding extreme long and short positions and thus on increasing the robustness with respect to the discrete trading. The model might not be better!

complete market	incomplete market
risk averse utility functions	...
non-constant interest rates	...
non-constant volatility	stochastic volatility
risk constraints (on the wealth)	convex constraints (on the strategy)
	Lévy noise

A risk constraint

We want to maximize the expected utility of the terminal wealth X_T^π **but also constrain the risk that the terminal wealth falls short of a benchmark**

$$q\,x_0, \quad q > 0\,.$$

The shortfall risk is measured by the expected loss in utility

$$\mathrm{E}[(U(X_T^\pi) - U(q\,x_0))^-]\,.$$

Optimization Problem: **For given** $x_0 > 0$, $q > 0$ **and** $\varepsilon > 0$ **solve**

$$\textbf{maximize} \quad \mathrm{E}[U(X_T^\pi)] \quad \textbf{s.t.} \quad \mathrm{E}[(U(X_T^\pi) - U(q\,x_0))^-] \le \varepsilon$$

over all admissible trading strategies π**.**

Or put a bound on

expected loss in wealth $\mathrm{E}[(X_T^\pi - q\,x_0)^-]$**,**

hedging price of the shortfall $\tilde{\mathrm{E}}[(X_T^\pi - q\,x_0)^-]$**.**

The optimal terminal wealth for bounded shortfall risk

For $x_0 > 0$ **and** y_1**,** $y_2 > 0$ **we define** $\underline{z} = \frac{1}{y_1} U'(q\, x_0)$**,** $\overline{z} = \frac{1+y_2}{y_1} U'(q\, x_0)$**,**

$$f(z; y_1, y_2) = \begin{cases} I(y_1 z), & z \in (0, \underline{z}], \\ q\, x_0, & z \in (\underline{z}, \overline{z}), \\ I(\frac{y_1}{1+y_2} z), & z \in [\overline{z}, \infty), \end{cases} \quad f'(z; y_1, y_2) = \begin{cases} y_1 I'(y_1 z), \\ 0, \\ \frac{y_1}{1+y_2} I'(\frac{y_1}{1+y_2} z). \end{cases}$$

Theorem 2 *If strictly positive and unique solutions* y_1, y_2 *of*

$$\mathrm{E}[\zeta_T f(\zeta_T; y_1, y_2)] = x_0, \quad \mathrm{E}[(U(f(\zeta_T; y_1, y_2)) - U(q\, x_0))^-] = \varepsilon$$

exist then the optimal terminal wealth is given by $\hat{X}_T = f(\zeta_T; y_1, y_2)$
Further, if $I'(y_1\, \zeta_T) \in L^r(\tilde{P})$ *for some* $r > 1$*, then for* $U_\alpha = x^\alpha/\alpha$

$$\frac{\hat{\pi}_t}{\hat{X}_t} = \frac{\sigma^{-2}}{1-\alpha} \Big(\tilde{\mathrm{E}}[C_{t,T} \,|\, \mathcal{E}_t, \hat{X}_t]\, B\, \eta_t + \tilde{\mathrm{E}}[C_{t,T} \int_t^T (\sigma D_t \mathcal{E}_{t,s}) b \sigma^{-2} dR_s \,|\, \mathcal{E}_t, \hat{X}_t] \Big)$$

where $\mathcal{E}_{t,s} = \mathcal{E}_s\, \zeta_t$*,* $\zeta_{t,T} = \zeta_T / \zeta_t$*,* $\quad C_{t,T} = g(\mathcal{E}_t, \zeta_{t,T}) - \zeta_{t,T}\, q\, x_0 / \hat{X}_t \mathbf{1}_{\{\underline{z}\zeta_t^{-1} < \zeta_{t,T} < \overline{z}\zeta_t^{-1}\}}$.

Remarks

- Due to the discretization and the dependency on $\hat{X}$, using the optimal trading strategies does not yield the desired properties (risk constraints are not fulfilled).
- It might help to compute the parameters of the optimal strategy at each time step again, based on the current wealth level. In contrast to the unconstrained case this is not optimal!
- Works similar for expected loss in wealth under P and $\tilde{P}$ (price of hedging strategy), $\varepsilon = 0$ (portfolio insurer). Difficult for general utility and VaR-constraint.

complete market	incomplete market
risk averse utility functions	...
non-constant interest rates	...
non-constant volatility	stochastic volatility
risk constraints (on the wealth)	convex constraints (on the strategy)
	Lévy noise

Convex Constraints

For n stocks convex constraints are of the form

$$f_t^\pi := \pi_t \,/\, X_t^\pi \in K, \quad t \in [0,T],$$

for a convex, closed $K \subseteq I\!R^n$ with $0 \in K$. Examples:

- **Constraints on short and long positions:**

$$K = \{x \in I\!R^n \,:\, l_i \leq x_i \leq u_i, i = 1, \ldots, n\}.$$

- **Maximum fraction u_0 of the wealth we can borrow from the bank:**

$$K = \{x \in I\!R^n \,:\, 1 - \mathbf{1}_n^\top x \geq u_0 \text{ and } x_i \geq 0, i = 1, \ldots, n\}.$$

- **No investment in stocks $m+1, \ldots, n$ (classical incomplete market):**

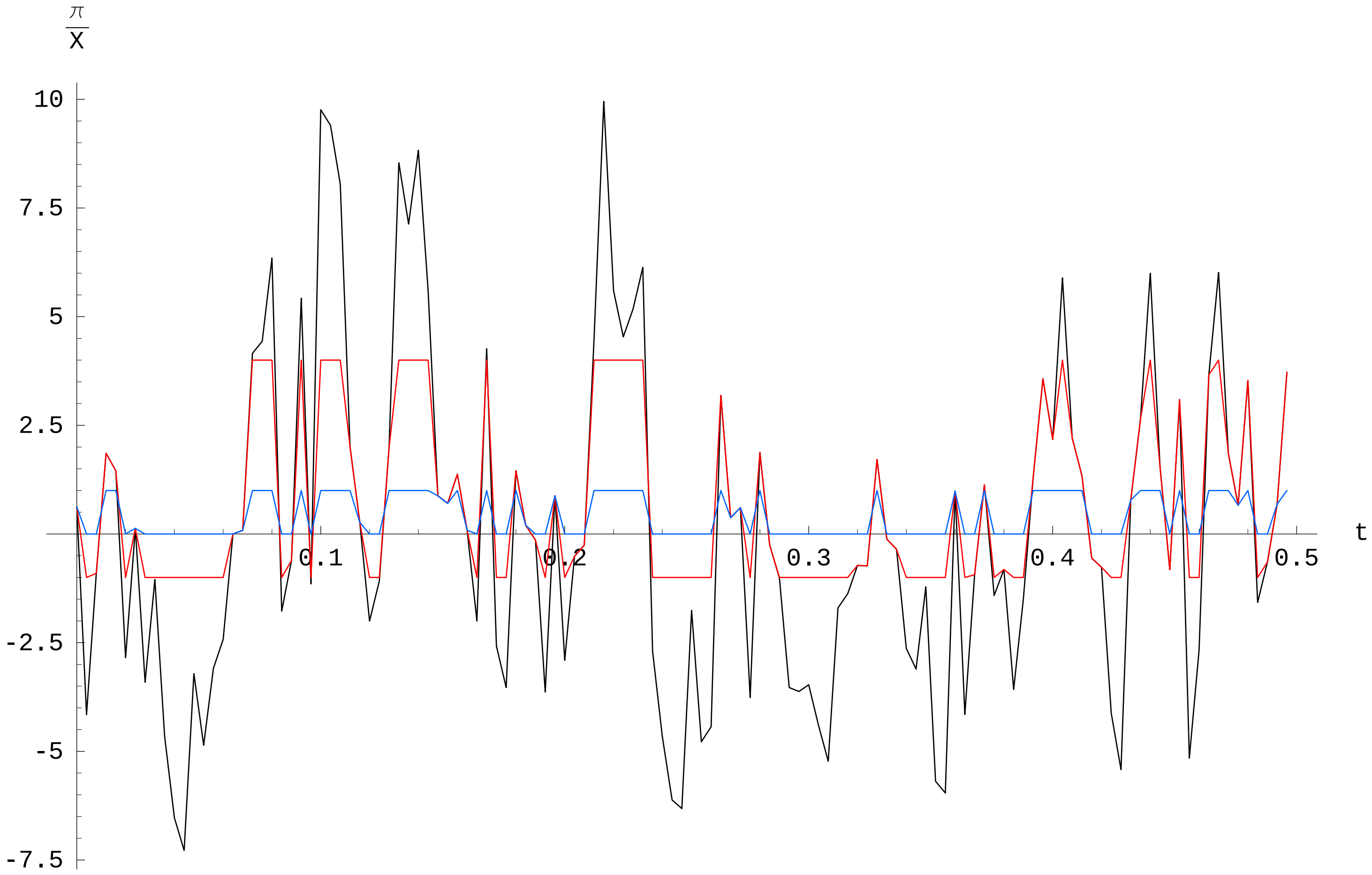
$$K = \{x \in I\!R^n \,:\, x_{m+1} = \cdots = x_n = 0\}.$$

Optimization Problem: For given $x_0 > 0$ and constraints K

$$\textbf{maximize} \quad \mathrm{E}[U(X_T^\pi)] \quad \textbf{subject to} \quad f^\pi \in K, \quad t \in [0,T]$$

over all admissible trading strategies π.

Optimal risky fractions for logarithmic utility



Optimal risky fractions for no constraints, K=[-1,4], K=[0,1].

Application to market data

We consider (as we did before) daily prices of 20 stocks of the DJII and the corresponding historic interest rates. With starting years 1972,...,1996 we estimate parameters in 5 years and apply the optimal strategies in the following year (for each stock, hence 500 experiments).

strategy	av. X_T	m. X_T	av. $\log(X_T)$	m. $\log(X_T)$	aborted
b/h	1.153	1.121	0.116	0.114	–
Merton	1.136	1.089	$-\infty$ (-0.016)	0.086	2
optimal, $K = \mathbb{R}$	1.520	1.145	$-\infty$ (0.042)	0.135	12
$K = [-1, 4]$	1.357	1.196	0.157	0.179	0
$K = [0, 1]$	1.172	1.156	0.141	0.145	0

Remarks

- Explicit strategies for general utility functions difficult to obtain.
- Easy for logarithmic utility, but non-trivial in the multidimensional case. If no natural restrictions apply (like no short-selling, no borrowing) the choice of the boundaries is quite arbitrary.
- Using one 'asset' as state process for the volatility process and using a constraint forbidding to invest in this asset, one can include stochastic volatility models in this setup. Strong requirements on the observability of the state process via the other assets are needed.

complete market	incomplete market
risk averse utility functions	...
non-constant interest rates	...
non-constant volatility	stochastic volatility
risk constraints (on the wealth)	convex constraints (on the strategy)
	Lévy noise

A hidden Markov model for stock returns with jumps

We now consider a stock with prices S and returns R given by

$$dS_t = S_{t-} dR_t \,, \quad dR_t = \mu_t \, dt + \sigma \, dW_t + \gamma \, d\tilde{N}_t \,, \quad t \in [0,T]$$

where $\gamma > -1$, and $\tilde{N}$ is a **compensated Poisson process**, independent of W and μ, with intensity λ, i.e. $\tilde{N}_t = N_t - \lambda t$.

Now a trading strategy $\pi = (\pi_t)_{t \in [0,T]}$ has to be $\mathcal{F}^R$**- predictable**.

The wealth process is given by $\quad dX_t^\pi = \pi_t dR_t = \pi_t(\mu_t \, dt + \sigma \, dW_t + \gamma \, d\tilde{N}_t)$.

Optimization Problem: Solve

$$V(x_0) = \sup \{ \, \mathrm{E}[U(X_T^\pi)] \mid \pi \quad \text{with } X^\pi \geq 0 \, \} \,,$$

i.e. find $\hat{\pi}$ with $\mathrm{E}[U(\hat{X}_T)] = V$, $\hat{X} \geq 0$.

HMM filtering and separation principle

In continuous time we can extract the continuous part

$$R_t^c = \int_0^t b^\top Y_s\, ds + \sigma\, W_t\ .$$

So we can use as reference measure $\tilde{P}$ defined analogously as above.

The innovation process V,

$$V_t = \tilde{W}_t - \int_0^t \sigma^{-1} b^\top \eta_s\, ds$$

is a Brownian motion under P with $\mathcal{F}^V = \mathcal{F}^{\tilde{W}} \subseteq \mathcal{F}^R$ and

$$\begin{aligned} dR_t &= b^\top Y_t + \sigma\, dW_t + \gamma\, d\tilde{N}_t \\ &= \sigma\, d\tilde{W}_t + \gamma\, d\tilde{N}_t \\ &= b^\top \eta_t\, dt + \sigma\, dV_t + \gamma\, d\tilde{N}_t\ . \end{aligned}$$

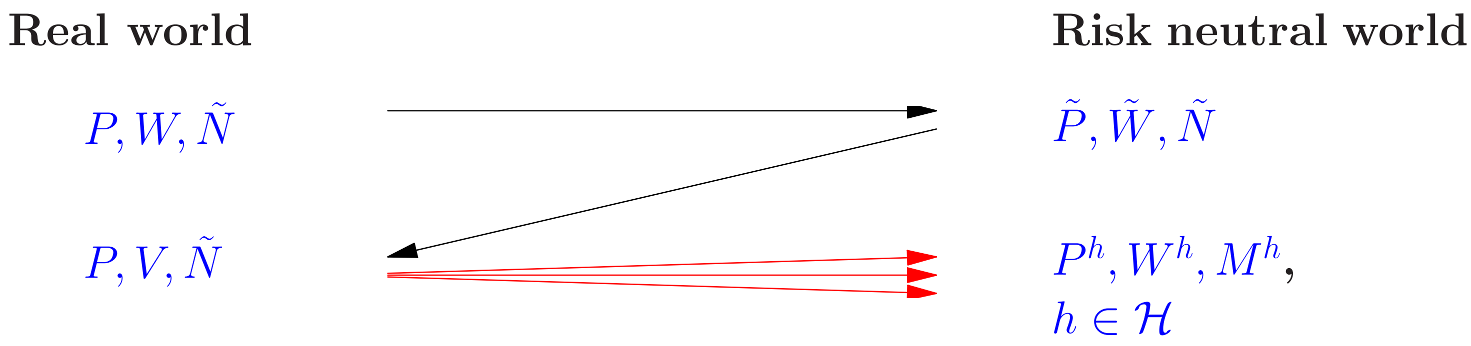
So by the transformations $\quad P, W \quad \longrightarrow \quad \tilde{P}, \tilde{W} \quad \longrightarrow \quad P, V$

we have an incomplete market model with full information.

Thus we follow the separation principle, see Genotte (1986):

Do the filtering first – and the optimization afterwards

Existence and verification result



Theorem 3 *(Kramkov and Schachermayer)*: *Suppose* $\sup_\pi \mathrm{E}[U(X_T^\pi)] < \infty$ *and* U *has asymptotic elasticity less than* 1. *Then an optimal solution exists.*

Lemma 1: *Suppose* $\hat{y} > 0$ *and a density* $\hat{Z}_T$ *exist such that*

(1) $\mathrm{E}[U(I(\hat{y}\,\hat{Z}_T))] < \infty$,

(2) $I(\hat{y}\,\hat{Z}_T)$ *can be hedged by* $\hat{\pi}$, *and*

(3) $\mathrm{E}[\hat{Z}_T\, I(\hat{y}\,\hat{Z}_T)] = x_0$.

Then $\hat{\pi}$ *is an optimal trading strategy and* $\hat{X}_T = I(\hat{y}\,\hat{Z}_T)$.

A good class of equivalent martingale measures

Denote by $\mathcal{H}$ the set of all real valued, bounded processes $h = (h_t)_{t\in[0,T]}$ which are predictable w.r.t. $\mathcal{F}^V$ and satisfy $h_t < 1$. For $h \in \mathcal{H}$ we define P^h by $\frac{dP^h}{dP} = Z_T^h$,

$$dZ_t^h = -Z_{t-}^h \sigma^{-1}(b^\top \eta_t - h_t \gamma\, \lambda)\, dV_t - Z_{t-}^h h_t dN_t\ .$$

Then $dR_t = \sigma\, dW_t^h + \gamma\, dM_t^h$, where

$$dW_t^h = dV_t + \sigma^{-1}(b^\top \eta_t - h_t \gamma\, \lambda) dt\ , \qquad dM_t^h = d\tilde{N}_t + h_t \lambda\, dt$$

define martingales under P^h.

Under logarithmic utility $X_t^\pi = x_0 / Z_t^h$. Comparing

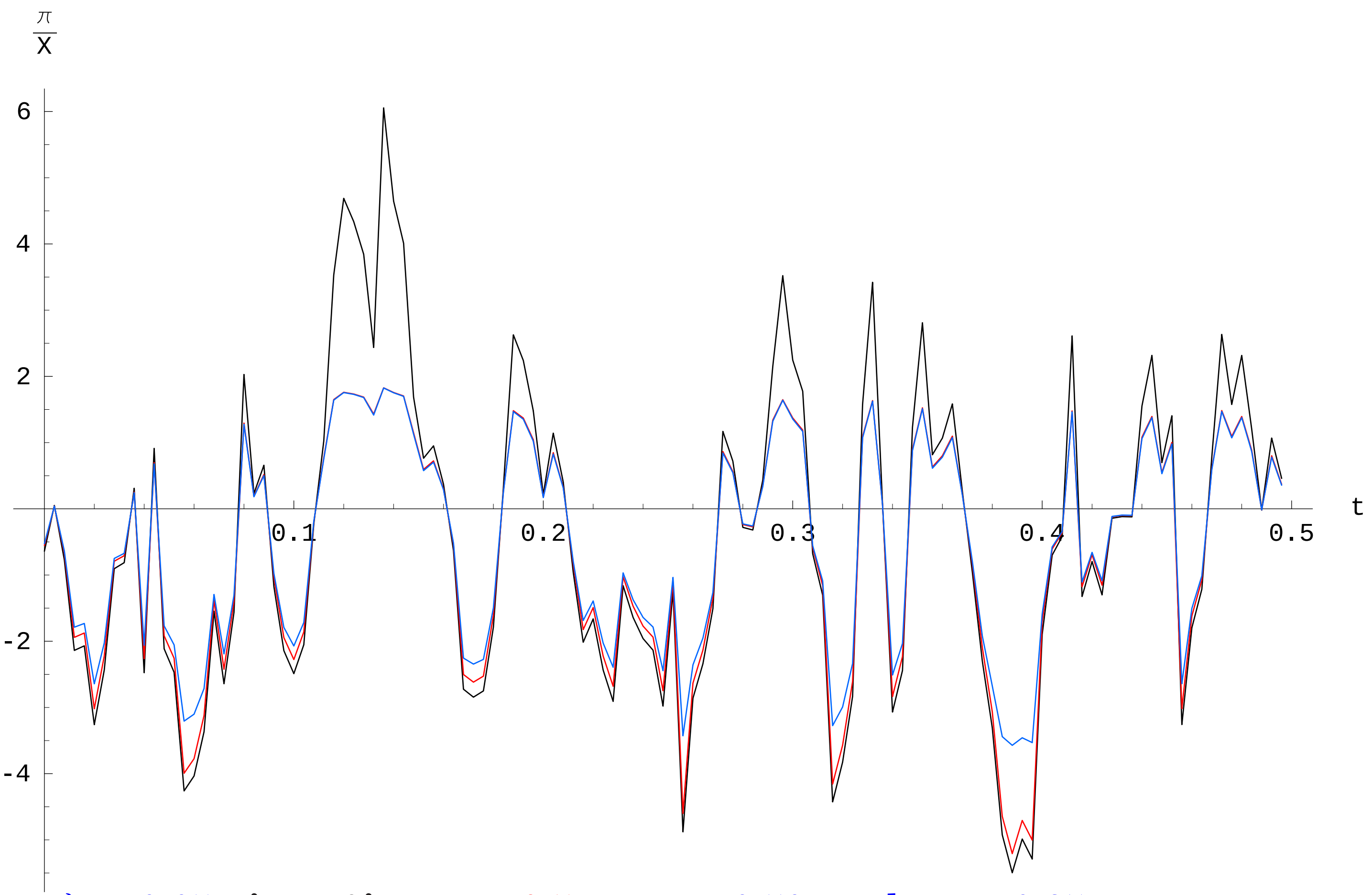
$$\begin{aligned} x_0 / Z_T^h &= x_0 + \int_0^T X_{t-} \sigma^{-1}(b^\top \eta_t - h_t \gamma\, \lambda) dW_t^h + \int_0^T X_{t-} \frac{h_t}{1-h_t} dM_t^h \\ X_T^\pi &= x_0 + \int_0^T \pi_t \sigma\, dW_t^h + \int_0^T \pi_t \gamma\, dM_t^h \end{aligned}$$

we get for the optimal risky fraction $f_t = \pi_t / X_{t-}^\pi$

$$f_t = \frac{b^\top \eta_t - h_t \gamma\, \lambda}{\sigma^2}, \qquad f_t \gamma = \frac{h_t}{1-h_t}\ .$$

Optimal trading strategies for two jump sizes γ_d, γ_u

One sample path for $\sigma = 0.4$, $r = 0.06$, $b = (0.8, -0.5)^\top$, $Q_{12} = 30$, $Q_{21} = 20$.



For $\lambda = 0.05$**:** '$\gamma = 0$', $\gamma = -0.5$, $\gamma_d = -0.50$ **and** $\gamma_u = 0.25$.

Application to market data

As above we consider daily prices of 20 stocks of the DJII and the corresponding historic interest rates. With starting years 1972,. . . ,1996 we estimate parameters in 5 years (assuming no jumps) and apply the optimal strategies in the following year (for each stock, hence 500 experiments).

jumps	strategy	av. X_T	m. X_T	av. $\log(X_T)$	m. $\log(X_T)$	aborted
0	optimal	1.569	1.113	$-\infty$ (0.006)	0.107	13
	Merton	1.140	1.087	$-\infty$ (-0.036)	0.084	3
	b/h	1.153	1.121	0.116	0.114	–
$\gamma = -\frac{1}{3}, +\frac{1}{3}$	optimal	1.311	1.170	0.154	0.157	0
$\lambda = 0.05$	Merton	1.128	1.098	0.045	0.093	0
	b/h	1.153	1.121	0.116	0.114	–

Remarks

- **Trading strategies for general utility functions are difficult to obtain.**
- **Works also for general Lévy noise, if set of jumpsizes is closed and does not contain 0. But in practice only big jumps (crashes) can be distinguished from the Brownian noise.**
- **Has a good performance and a good interpretation as modeling of crashes and sudden increases in the stock prices). Only, if the next jump is bigger then the anticipated worst case, then ... .**

complete market	incomplete market
risk averse utility functions	...
non-constant interest rates	...
non-constant volatility	stochastic volatility
risk constraints (on the wealth)	convex constraints (on the strategy)
	Lévy noise

General remarks

- Alternative models for μ considered under partial information are those with linear Gaussian dynamics (LGD)

$$d\mu_t = \alpha(\delta - \mu_t)dt + \beta d\bar{W}_t$$

  or simply a time-independent distribution.

- Parameters (asides from the general Lévy case) can be estimated using the expectation maximization (EM) algorithm or Markov chain Monte Carlo (MCMC) methods.

- We are still working on explicit representations of the optimal trading strategies for power utility under convex constraints or Lévy noise.

- Conclusion: Many possible improvements which yield 'quite robust', well performing strategies. But optimal discrete time strategies, robustness, taxes, transaction costs, large portfolios have still to be looked at.

A rather incomplete list of useful references

Full information:

- Merton (1969): For $U_\alpha(x) = x^\alpha/\alpha$, constant μ_0, no jumps: $\hat{\pi}_t/\hat{X}_t = \frac{1}{1-\alpha}\frac{\mu_0}{\sigma^2}$.
- Ocone and Karatzas (1991): Full information, Malliavin derivative
- Kramkov and Schachermayer (1999): Utility maximization in an incomplete market

Partial information:

- Karatzas and Xue (1991): Partial information, optimal consumption
- Kuwana (1995): Logarithmic utility
- Lakner (1995, 1998): Optimal terminal wealth, Kalman case (LGD)
- Pham and Quenez (2001): Incomplete market with stochastic volatility
- Elliott (1994), James, Krishnamurthy and Le Gland (1996): (robust) HMM filtering
- Sass and Haussmann (2004): μ as a continuous time Markov chain (CTMC)
- Rishel (1999), Bäuerle and Rieder (2005): HJBs for LGD and CTMC

Non-constant and stochastic volatility

- Hobson and Rogers (1998): Complete models with stochastic volatility
- Pham and Quenez (2001): Incomplete market with stochastic volatility

- **Haussmann and Sass (2004): Non-constant volatility, μ CTMC**

Risk constraints:

- **Basak and Shapiro (2001): Constant μ, value-at-risk based utility max.**
- **Emmer, Klüppelberg, and Korn (2001): Constant μ, capital at risk based**
- **Gabih and Wunderlich (2005): Const. μ, utility max. under risk constraints**
- **Gabih, Sass, and Wunderlich (2005): μ as CTMC, loss in utility**

Convex constraints:

- **Cvitanić (1997): Overview convex constraints**
- **Sass (2005): Convex constraints with μ as CTMC**

Lévy noise:

- **Kallsen (2000), Emmer and Kl"uppelberg (2004): Optimization for Lévy processes (constant μ)**
- **Cadenillas (2002): Stochastic Maximum principle**
- **Bäuerle and Rieder (2005): Constant μ, intensity as CTMC**
- **Hausmann and Sass (2005): μ as CTMC under Lévy noise, working paper**
- **Runggaldier (2005): μ with LGD under Poisson noise**