

# DYNAMIC CDO TERM STRUCTURE MODELLING

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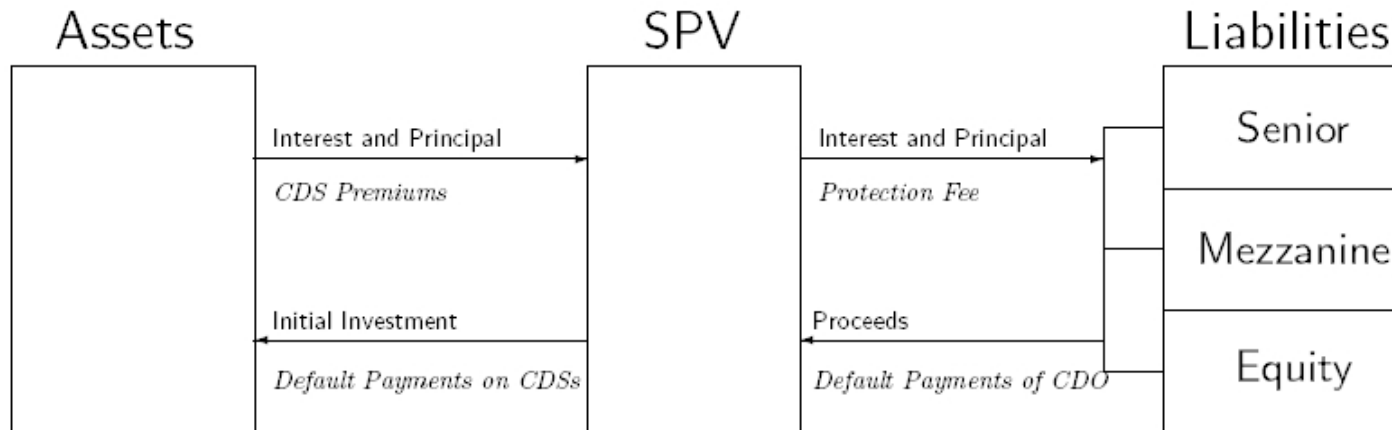
## Overview

1. **Collateralized Debt Obligations (CDOs)**
2.  $(T, x)$ -Bonds
3. Arbitrage-free Term Structure Movements
4. Doubly Stochastic Framework

## Collateralized Debt Obligations (CDOs)

- most important type of portfolio credit derivative
- security backed by pool of reference entities (**assets**): bonds, loans, protection seller position in single name CDS, etc.
- assets sold to **special-purpose vehicle (SPV)**
- SPV issues notes on **CDO tranches** (**liabilities**)
- important reference indices: ITraxx Europe, CDX (USA),

## Basic Structure of a CDO



Payments in a CDO structure.

Payments corresponding to synthetic CDOs are in *italics*.

## Literature

Bennani (05): “The forward loss model: A dynamic term structure approach for the pricing of portfolio credit derivatives”

Cont and Minca (08), “Recovering portfolio default intensities implied by CDO quotes”

Ehlers and Schönbucher (06), “Pricing Interest Rate-Sensitive Credit Portfolio Derivatives”

Ehlers and Schönbucher (06), “Background Filtrations and Canonical Loss Processes for Top-Down Models of Portfolio Credit Risk”

Filipović, Overbeck and Schmidt (08), “Dynamic CDO Term Structure Modelling”

Schönbucher (05), “Portfolio losses and the term structure of loss transition rates: A new methodology for the pricing of portfolio credit derivatives”

Sidenius, Piterbarg and Andersen (IJTAF 08), “A new framework for dynamic credit portfolio loss modelling”

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## $(T, x)$ -Bonds

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ ,  $\mathbb{Q}$  risk-neutral measure
- CDO pool of credits normalized to 1.
- **Loss process**  $L_t = \sum_{s \leq t} \Delta L_s$   $[0, 1]$ -valued increasing MPP with abs. continuous compensator  $\nu(t, dx) dt$ .
- **$(T, x)$ -bond** pays  $1_{\{L_T \leq x\}}$  at maturity  $T$ ,  $x \in [0, 1]$ .  
Its price  $P(t, T, x)$  at  $t \leq T$  is decreasing in  $T$ , increasing in  $x$ .  
Note:  $P(t, T) = P(t, T, 1)$  is risk-free zero-coupon bond.

## Default Times of the $(T, x)$ -Bonds

**Lemma 1:** For any  $x \in [0, 1]$ , the process  $\mathbf{1}_{\{L_t \leq x\}}$  has **intensity**

$$\lambda(t, x) = \nu(t, (x - L_t, 1]).$$

That is,

$$M_t^x = \mathbf{1}_{\{L_t \leq x\}} + \int_0^t \mathbf{1}_{\{L_s \leq x\}} \lambda(s, x) ds \quad \text{is a martingale.}$$

Conversely,  $\lambda(t, x)$  uniquely determines  $\nu(t, dx)$  via

$$\nu(t, (0, x]) = \lambda(t, L_t) - \lambda(t, L_t + x), \quad x \in [0, 1].$$

*Proof.*  $F(L_t) - \int_0^t \int_0^1 (F(L_s + y) - F(L_s)) \nu(s, dy) ds$  is a martingale, for any bounded measurable function  $F$ .

For  $F(L_t) = \mathbf{1}_{\{L_t \leq x\}}$ , we obtain  $F(L_s + y) - F(L_s) = -\mathbf{1}_{\{L_s + y > x\}} \mathbf{1}_{\{L_s \leq x\}}$ . □

## $(T, x)$ -Bonds

- Contingent claim with payoff  $F(L_T)$  at  $T$  can be decomposed

$$F(L_T) = F(1) - \int_0^1 F'(y) \mathbf{1}_{\{L_T \leq y\}} dy$$

- Hence static replicating portfolio, at  $t \leq T$ , is

$$F(1)P(t, T) - \int_0^1 F'(y)P(t, T, y) dy$$

$\Rightarrow$   $(T, x)$ -bonds span all European type contingent claims

## Single Tranche CDOs (STCDOs)

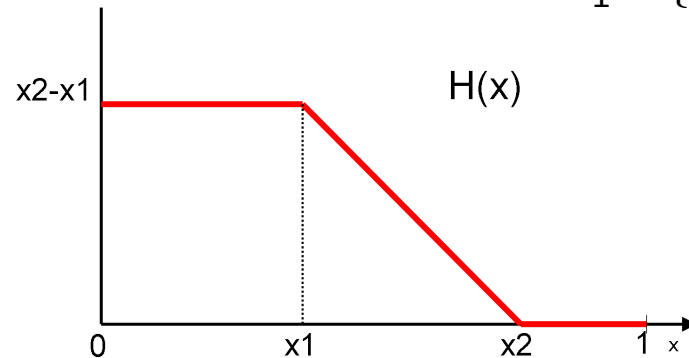
Standard instrument for investing in CDO-pool (e.g. iTraxx).

Specified by

- a number of future dates  $T_0 < T_1 < \dots < T_n$ ,
- a **tranche** with lower and upper detachment points  $x_1 < x_2$ ,
- a fixed **spread**  $S$ .

## Single Tranche CDOs (STCDOs)

Write  $H(x) = (x_2 - x)^+ - (x_1 - x)^+ = \int_{x_1}^{x_2} 1_{\{x \leq y\}} dy$



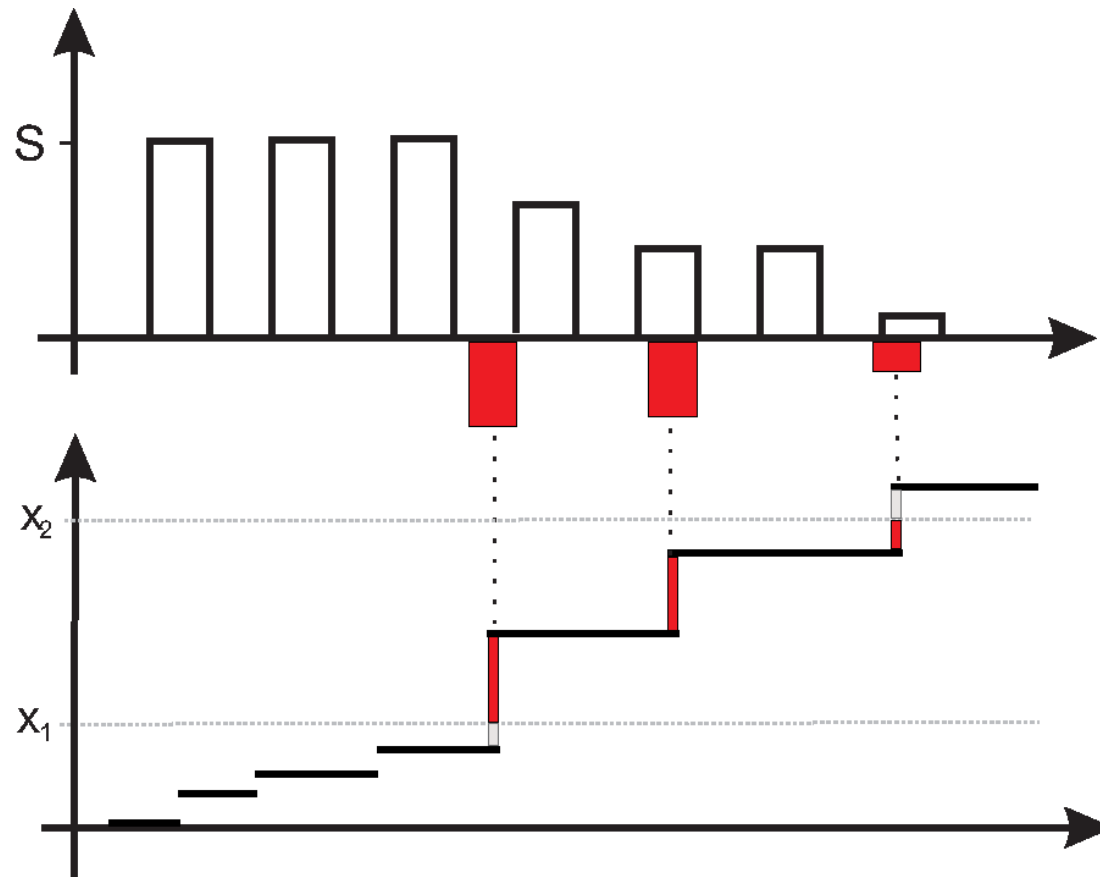
An investor in this STCDO

- **receives**  $SH(L_{T_i})$  at  $T_i$ ,  $i = 1, \dots, n$  (**payment leg**),
- **pays**  $-\Delta H(L_t) = H(L_{t-}) - H(L_t)$  at any time  $t \in (T_0, T_n]$  where  $\Delta L_t \neq 0$  (**default leg**).

$\Rightarrow$  STCDO can be priced via  $(T, x)$ -bonds

## Single Tranche CDOs (STCDOs)

Cash-flow attributed to tranche  $(x_1, x_2]$ :



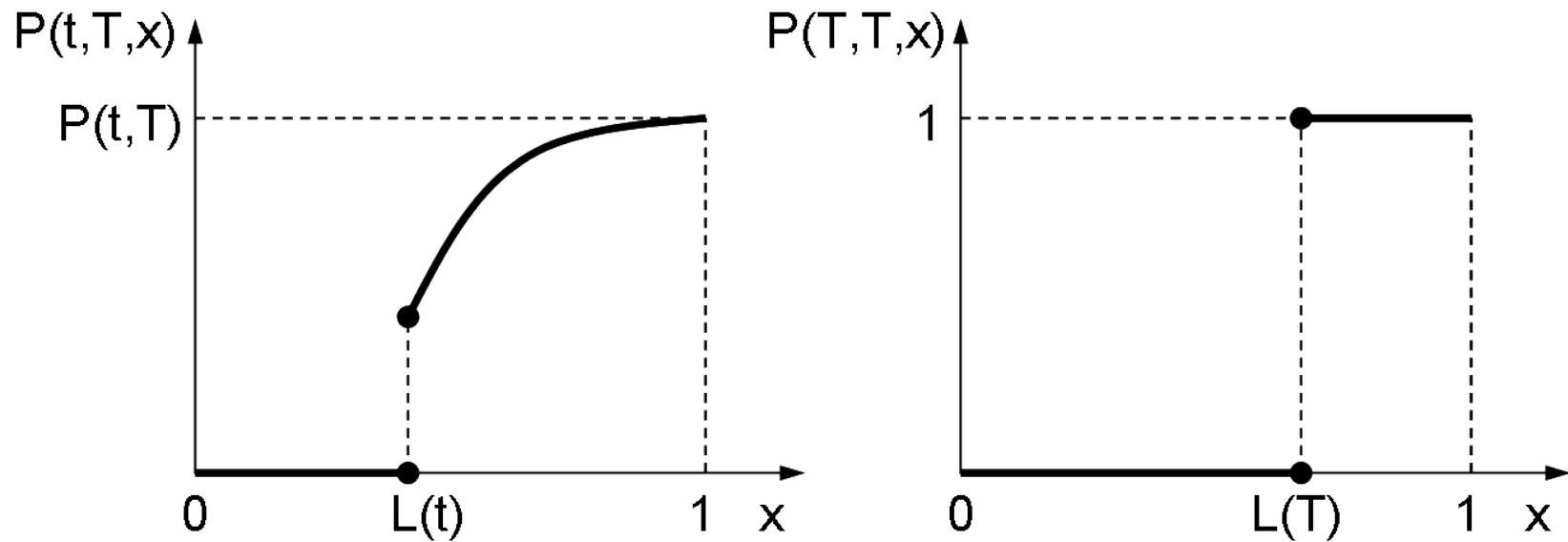
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## Term Structure Movements

**Aim:** describe  $(T, x)$ -bond price movements explicitly by

$$P(t, T, x) = \mathbf{1}_{\{L_t \leq x\}} e^{-\int_t^T f(t, u, x) du}$$

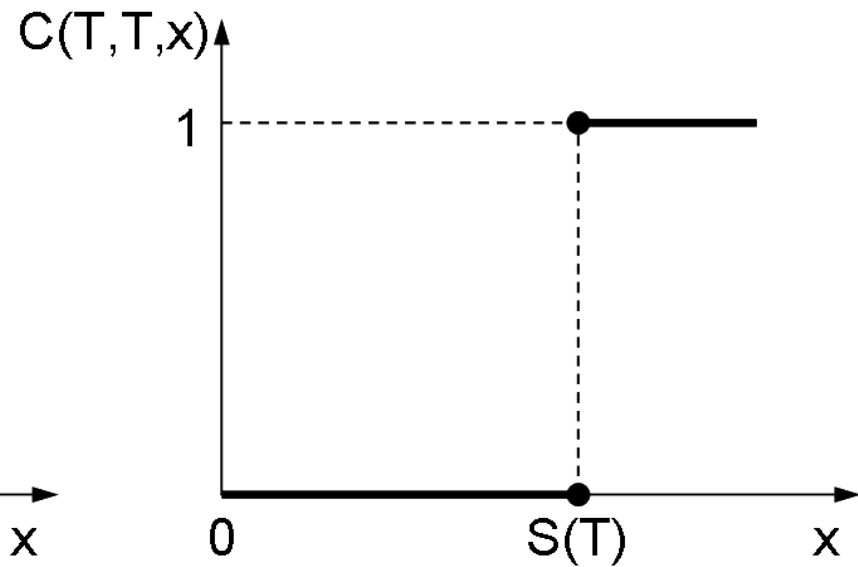
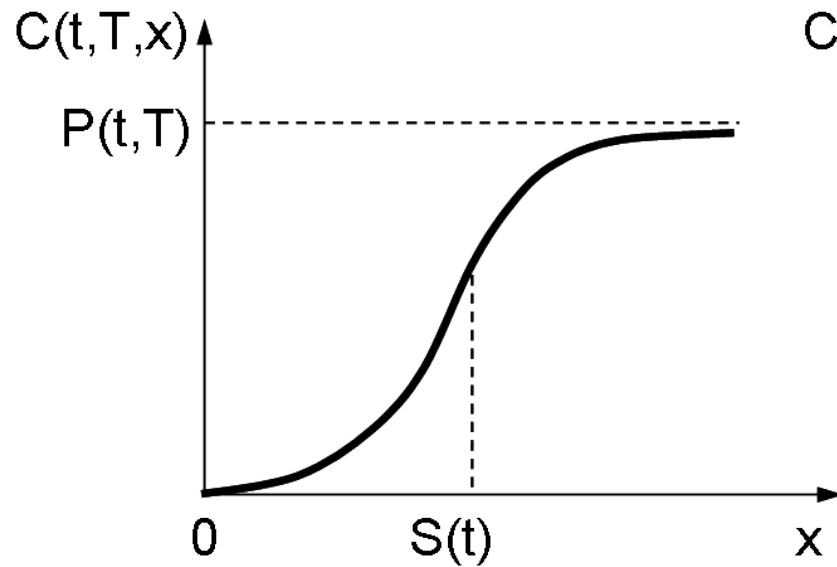


$P(t, T, x) = P(t, T) \mathbb{Q}^T [L_T \leq x \mid \mathcal{F}_t]$  is  $\mathcal{F}_t$ -conditional CDF of  $L_T$  w.r.t.  $\mathbb{Q}^T$

## Note the Difference

$\mathcal{F}_t$ -conditional CDF of stock price  $S_T$  w.r.t.  $\mathbb{Q}^T$

$$C(t, T, x) = P(t, T) \mathbb{Q}^T [S_T \leq x \mid \mathcal{F}_t]$$



## Term Structure Movements

$(T, x)$ -bond price

$$P(t, T, x) = \mathbf{1}_{\{L_t \leq x\}} e^{-\int_t^T f(t, u, x) du}$$

where  $f(t, T, x)$  is the  **$(T, x)$ -forward rate**

$$f(t, T, x) = f(0, T, x) + \int_0^t a(s, T, x) ds + \int_0^t b(s, T, x)^\top \cdot dW_s$$

Risk-free  **$T$ -forward rate**  $f(t, T) = f(t, T, 1)$

**short rate**  $r_t = f(t, t, 1)$

## Term Structure Movements

Include **contagion**:

- direct:  $\Delta f(t, T, x) = c(t, T, x; \Delta L_t)$
- indirect:  $b(t, T, x) = b(t, T, x; L)$ , same for  $a, c$

$$f(t, T, x) = f(0, T, x) + \int_0^t a(s, T, x; \mathbf{L}) ds + \int_0^t b(s, T, x; \mathbf{L})^\top \cdot dW_s \\ + \sum_{s \leq t} c(s, T, x; \Delta L_s) 1_{\{\Delta L_s > 0\}}$$

## Arbitrage-free Term Structure Movements

**No arbitrage (NA):**  $e^{-\int_0^t r_s ds} P(t, T, x)$  local martingale  $\forall (T, x)$

**Theorem 2:** NA is equivalent to

$$\int_t^T a(t, u, x) du = \frac{1}{2} \left\| \int_t^T b(t, u, x) du \right\|^2 + \int_0^1 \left( e^{-\int_t^T c(t, u, x; y) du} - 1 \right) \nu(t, dy),$$
$$\lambda(t, x) = f(t, t, x) - r_t$$

on  $\{L_t \leq x\}$ ,  $dt \otimes d\mathbb{Q}$ -a.s. for all  $(T, x)$ .

**NB:** recall  $\nu(t, dy) = -\lambda(t, L_t + dy) = -f(t, t, L_t + dy)$

## Single Tranche CDOs (STCDOs)

Write  $p(t, T, x) = e^{-\int_t^T f(t, u, x) du}$ .

**Lemma 4:** The value of the STCDO at time  $t \leq T_0$  is

$$\begin{aligned} & \Gamma(t, S) \\ &= \int_{(x_1, x_2]} \mathbf{1}_{\{L_t \leq y\}} \left( S \sum_{i=1}^n p(t, T_i, y) - p(t, T_0, y) + p(t, T_n, y) + \gamma(t, y) \right) dy \end{aligned}$$

where

$$\gamma(t, y) = \int_{T_0}^{T_n} f(t, u) p(t, u, y) du \quad \text{if } f(t, u) \text{ and } L_t \text{ are independent.}$$

**Forward STCDO spread**  $S^*(t)$  defined by  $\Gamma(t, S^*(t)) = 0$ .

**STCDO swaption** with strike  $K$  has payoff at maturity  $T_0$

$$\left( \sum_{i=1}^n \int_{(x_1, x_2]} \mathbf{1}_{\{L_t \leq y\}} p(T_0, T_i, y) dy \right) (K - S_{T_0}^*)^+ .$$

## Martingale Problem

**Aim:** exogenous specification of  $b(t, T, x)$  and  $c(t, T, x)$  determines full  $(T, x)$ -bond model  $P(t, T, x)$ .

**Martingale problem:** implicit loss process  $L_t$  such that  $\nu(t, dx) = -f(t, t, L_t + dx)$  becomes compensator

**Assumption:** canonical stochastic basis

$\Omega = \Omega_1 \times \Omega_2$ ,  $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$ ,  $\mathbb{Q}(d\omega_1, d\omega_2) = \mathbb{Q}_1(d\omega_1)\mathbb{Q}_2(\omega_1, d\omega_2)$ :

- $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$  carrying market information, i.e. Brownian motion  $W(\omega) = W(\omega_1)$ ,
- $(\Omega_2, \mathcal{H})$  canonical space of  $[0, 1]$ -valued increasing MPPs, loss process = coordinate process:  $L_t(\omega) = \omega_2(t)$
- $\mathbb{Q}_2$  probability kernel from  $\Omega_1$  to  $\mathcal{H}$  to be determined below.

## Martingale Problem

**Solution:** Jacod (75), “Multivariate Point Processes: Predictable Projection, Radon-Nikodym Derivatives, Representation of Martingales”

**Theorem 3:** Given vola and contagion parameters  $b(\omega; t, T, x) = b(\omega_1, \omega_2; t, T, x)$  and  $c(\omega; t, T, x, y) = c(\omega_1, \omega_2; t, T, x, y)$

1. Define  $a(t, T, x)$  via NA drift condition.
2. Solve for  $f(t, T, x)$  along **any loss path**  $\omega_2$ .
3. Jacod (75):  $\exists$  unique kernel  $\mathbb{Q}_2$  such that NA holds.

## Martingale Problem

**Theorem 3 contd.:** Moreover, on  $\{\tau_n < \infty\}$ ,

$$\mathbb{Q} \left[ \tau_{n+1} - \tau_n > t \mid \mathcal{G} \otimes \mathcal{H}_{\tau_n} \right] = e^{-\int_{\tau_n}^{\tau_n+t} \nu(\omega_1, \omega_2(\tau_n); s, [0,1]) ds}$$

and

$$\mathbb{Q} [\Delta L_{\tau_n} \in A \mid \mathcal{G} \otimes \mathcal{H}_{\tau_n-}] = \frac{\nu(\tau_n, A)}{\nu(\tau_n, [0, 1])}, \quad A \subset [0, 1]$$

where  $0 < \tau_1 < \tau_2 < \dots$  denote jump times of  $L$ .

## Monte-Carlo algorithm

Along any Brownian path  $\omega_1^{(1)}, \dots, \omega_1^{(N)}$ , by recursion

- solve  $f(t, T, x)$  with  $L_t \equiv L_{\tau_{j-1}}$  for  $t \geq \tau_{j-1}$
- set  $\tau_j = \inf \left\{ t \mid \int_{\tau_{j-1}}^t \lambda(s, L_{\tau_{j-1}}) ds \geq \epsilon^{(j)} \right\}$ ,  $\epsilon^{(j)} \sim \text{exp iid}$
- simulate  $\Delta L_{\tau_j} \sim \frac{-\lambda(\tau_j, L_{\tau_{j-1}} + dx)}{\lambda(\tau_j, L_{\tau_{j-1}})}$ ,  $x \geq 0$
- restart at  $\tau_j$  with  $\Delta f(\tau_j, T, x) = c(\tau_j, T, x; \Delta L_{\tau_j})$

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## Doubly Stochastic Framework

No contagion  $b(\omega) = b(\omega_1)$  and  $c = 0$ .

Then  $L$  becomes (uniquely)  $\mathcal{G}$ -conditional Markov.

Moreover, for any  $\mathcal{G}$ -measurable  $X \geq 0$ :

$$\mathbb{E}[X \mathbf{1}_{\{L_T \leq x\}} \mid \mathcal{F}_t] = \mathbf{1}_{\{L_t \leq x\}} \mathbb{E} \left[ X e^{-\int_t^T \lambda(s,x) ds} \mid \mathcal{G}_t \right].$$

(This is the SPA 08 framework)

## Affine Term Structure Models

State space  $\mathcal{Z} \subset \mathbb{R}^d$ , state process

$$\begin{aligned}dZ_t &= \mu(Z_t)dt + \sigma(Z_t) \cdot dW_t, \\ Z_0 &= z\end{aligned}$$

### Affine term structure (ATS)

$$f(t, T, x) = A'(t, T, x) + B'(t, T, x)^\top \cdot Z_t$$

Write  $A(t, T, x) = \int_t^T A'(t, u, x)du$ ,  $B(t, T, x) = \int_t^T B'(t, u, x)du$ .

## Affine Term Structure Models

**Theorem 6:** Suppose ATS and NA holds for all  $z \in \mathcal{Z}$ . Then (“generically”)  $Z$  is **affine**:

$$\mu(z) = \mu_0 + \sum_{i=1}^d z_i \mu_i, \quad \frac{1}{2} \sigma \cdot \sigma^\top(z) = \nu_0 + \sum_{i=1}^d z_i \nu_i$$

and  $A$  and  $B$  solve **Riccati equations**, for  $t \leq T$ ,

$$\begin{aligned} -\partial_t A(t, T, x) &= \mathbf{A}'(t, t, \mathbf{x}) + \mu_0^\top \cdot B(t, T, x) - B(t, T, x)^\top \cdot \nu_0 \cdot B(t, T, x) \\ -\partial_t B_i(t, T, x) &= \mathbf{B}'_i(t, t, \mathbf{x}) + \mu_i^\top \cdot B(t, T, x) - B(t, T, x)^\top \cdot \nu_i \cdot B(t, T, x) \end{aligned}$$

with  $A(T, T, x) = 0$  and  $B(T, T, x) = 0 \forall (T, x)$ .

## Affine Term Structure Models

**Theorem 7:** Conversely, suppose  $Z$  is affine, and let  $A'(t, t, \mathbf{x})$ ,  $B'(t, t, \mathbf{x})$  be bounded functions such that  $A'(t, t, \mathbf{x}) + B'(t, t, \mathbf{x})^\top \cdot z$  is decreasing and càdlàg in  $x$  for all  $t$  and  $z \in \mathcal{Z}$ .

Let  $A$  and  $B$  be given as solutions of the **Riccati equations**.  
Then

$$P(t, T, x) = 1_{\{L_t \leq x\}} e^{-A(t, T, x) - B(t, T, x)^\top \cdot Z_t}$$

defines an arbitrage-free  $(T, x)$ -bond market.

## Affine Term Structure Models

**Simple example:**  $dZ_t = (\mu_0 + \mu_1 Z_t)dt + \sigma\sqrt{Z_t}dW_t$ .

Moreover:  $\mathbf{A}'(t, t, \mathbf{x}) = \alpha(t, \mathbf{x})$  with  $\alpha(t, \mathbf{1}) \equiv \mathbf{r} \geq 0$ ,  $\mathbf{B}'(t, t, \mathbf{x}) = \beta(\mathbf{x})$  with  $\beta(\mathbf{1}) \equiv \mathbf{0}$ , so that:

$$r_t \equiv \mathbf{r}, \quad \text{and} \quad \lambda(t, x) = \alpha(t, \mathbf{x}) - \mathbf{r} + \beta(\mathbf{x})Z_t.$$

The **Riccati equations** become

$$A(t, T, x) = \int_t^T (\alpha(s, \mathbf{x}) + \mu_0 B(s, T, x)) ds$$
$$-\partial_t B(t, T, x) = \beta(\mathbf{x}) + \mu_1 B(t, T, x) - \frac{\sigma^2}{2} B(t, T, x)^2, \quad B(T, T, x) = 0$$

with solution

$$B(t, T, x) \equiv B(T-t, x) = \frac{2\beta(\mathbf{x}) \left( e^{\rho(\mathbf{x})(T-t)} - 1 \right)}{\rho(\mathbf{x}) \left( e^{\rho(\mathbf{x})(T-t)} + 1 \right) - \mu_1 \left( e^{\rho(\mathbf{x})(T-t)} - 1 \right)}$$

where  $\rho(\mathbf{x}) = \sqrt{\mu_1^2 + 2\sigma^2\beta(\mathbf{x})}$ .

We obtain

$$f(t, T) \equiv \mathbf{r}$$

$$f(t, T, x) = \alpha(\mathbf{T}, \mathbf{x}) + \mu_0 B(T-t, x) + \partial_T B(T-t, x) Z_t.$$

→ efficient computation of STCDO values and swaptions

→ matches any initial spread curve  $f(0, T, x)$  by choice of  $\alpha(\mathbf{T}, \mathbf{x})$ .