

# **Strong Taylor approximation of SDEs and application to the Lévy LIBOR model**

Antonis Papapantoleon

FAM – TU Vienna

Joint work with Maria Siopacha and Friedrich Hubalek

PRisMa 2008: One-Day Workshop on Portfolio Risk Management

TU Vienna, Austria; 29 September 2008

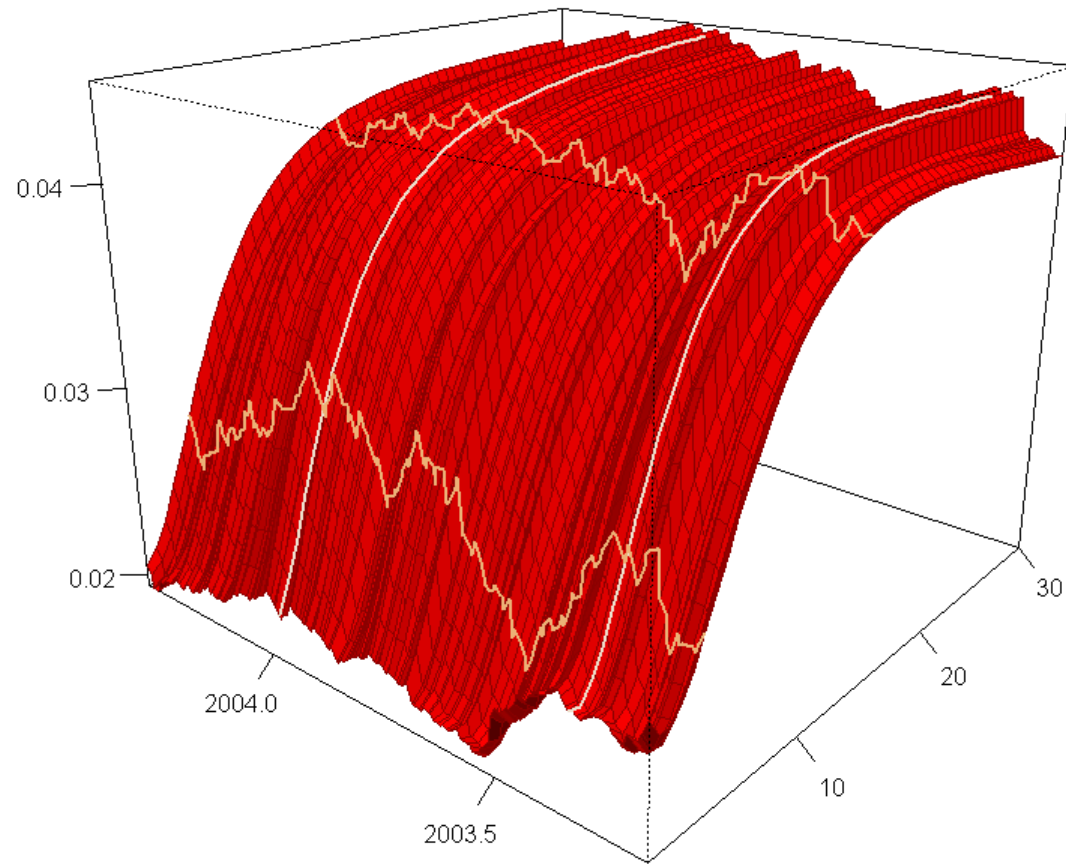
## Derivatives markets

According to the BIS Quarterly Review, the notional amounts outstanding for OTC derivatives in billions of US\$, were:

	Dec 2006	Jun 2007	Dec 2007
Foreign exchange	40,239	48,620	56,238
Interest rate	291,115	346,937	393,138
Equity-linked	7,488	9,202	8,509
Commodity	7,115	7,567	9,000
Credit default swaps	28,650	42,580	57,894
Unallocated	39,682	61,501	71,225
Total	414,290	516,407	596,004

- Interest rate derivatives are an important part of global financial markets

# Interest rates evolution



Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)

## Basic interest rates – Notation

$B(t, T)$  time- $t$  price of a zero coupon bond for maturity  $T$ , i.e.  $B(T, T) = 1$ ;

$L(t, T)$  time- $t$  forward LIBOR for  $[T, T + \delta]$ :

$$L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)$$

$F(t, T, U)$  time- $t$  forward price for maturities  $T$  and  $U$ :  $F(t, T, U) = \frac{B(t, T)}{B(t, U)}$

**Relationships:**

$$F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T)$$

(1)

## Modeling approaches

Short rate models, Forward rate models (HJM), . . .

**LIBOR market models:** LIBOR rates are modeled directly as

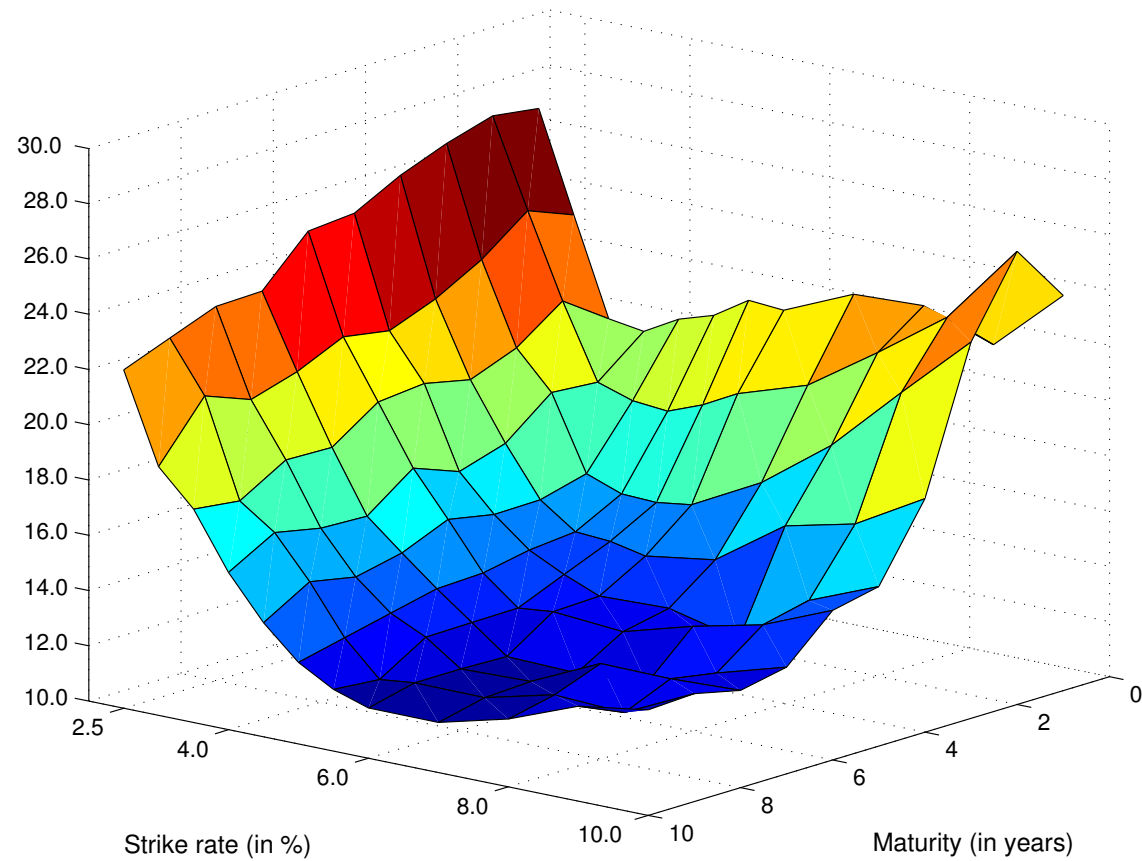
$$L(t, T_k) = L(0, T_k) \mathcal{E} \left( \int_0^{\cdot} \lambda(s, T_k) dW_s^{T_{k+1}} \right)_t$$

for **all**  $T \in \{T_0, T_1, T_2, \dots, T_{N-1}, T_N\}$ , a *discrete* tenor structure. Here  $W^{T_{k+1}}$  is a Brownian motion under the *forward martingale measure*  $\mathbb{P}_{T_{k+1}}$ . Forward measures are related via

$$\frac{d\mathbb{P}_{T_{k+1}}}{d\mathbb{P}_{T_k}} = \frac{F(T_k, T_k, T_{k+1})}{F(0, T_k, T_{k+1})}.$$

- **Advantages:** **positive** LIBOR rates; consistent with market practice i.e. Black 1976.  
Calibration? Tractability?

# Calibration problems



- ▶ Caplet implied volatilities are constant neither across strike nor across maturity!

# Time-inhomogeneous Lévy processes

Let  $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$  be a complete stochastic basis, where  $\mathcal{F} = \mathcal{F}_{T_*}$  and  $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T_*]}$ .

A **time-inhomogeneous Lévy process**, is an adapted, càdlàg real-valued stochastic process  $H = (H_t)_{0 \leq t \leq T_*}$  with  $H_0 = 0$  a.s., such that:

**(D1)**  $H$  has *independent increments*, i.e.  $H_t - H_s$  is independent of  $\mathcal{F}_s$ ,  $0 \leq s < t \leq T_*$ ,

**(D2)** the *law* of  $H_t$  is described by the characteristic function

$$\mathbb{E} [e^{iuH_t}] = \exp \left( \int_0^t \kappa_s(iu) ds \right), \quad (2)$$

$$\kappa_s(iu) = iub_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) F_s(dx), \quad (3)$$

for all  $t \in [0, T_*]$ . Here  $b_s \in \mathbb{R}$ ,  $c_s \in \mathbb{R}_{\geq 0}$  and  $F_s$  are Lévy measures,  $s \in [0, T_*]$ .

**Assumption (AC).** The triplets  $(b_s, c_s, F_s)$  satisfy:

$$\int_0^{T_*} \left( |b_s| + c_s + \int_{\mathbb{R}} (1 \wedge x^2) F_s(dx) \right) ds < \infty.$$

►  $H$  is a semimartingale on  $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ . [FTAP applies]

**Assumption (EM).** There exist  $M, \varepsilon > 0$ , such that the Lévy measures  $F_s$  satisfy:

$$\int_0^{T_*} \int_{\{|x|>1\}} e^{ux} F_s(dx) ds < \infty, \quad \forall u \in [-(1 + \varepsilon)M, (1 + \varepsilon)M] =: \mathbb{M}.$$

Moreover,  $\int_{\{|x|>1\}} e^{ux} F_s(dx) < \infty$  for all  $s \in [0, T_*]$  and all  $u \in \mathbb{M}$ .

►  $\kappa_s$  can be extended to  $\mathbb{R} \times i\mathbb{M}$ , for all  $s \in [0, T_*]$ . [Option pricing and Greeks]

## Semimartingale semantics

The process  $H$  is a *special* semimartingale with **canonical decomposition**

$$H_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu - \nu)(ds, dx) \quad (4)$$

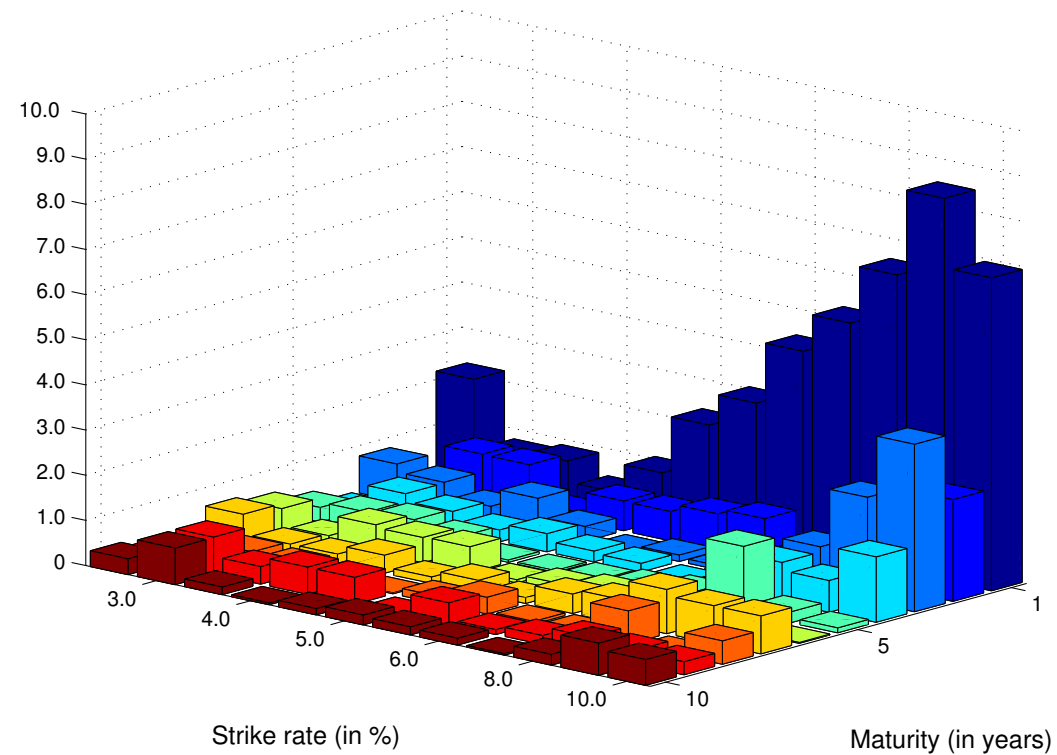
and the triplet of **predictable characteristics** is  $(B, C, \nu)$ , where

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0, t] \times A) = \int_0^t \int_A F_s(dx) ds, \quad (5)$$

$A \in \mathcal{B}(\mathbb{R})$ .  $W$  is a  $\mathbb{P}$ -Brownian motion,  $\mu$  is the random measure of jumps of  $H$  and  $\nu$  is the compensator of  $\mu$ .

►  $(B, C, \nu)$  completely determines the *distribution* of  $H$ .

# Calibration results



- ▶ Absolute errors in caplet calibration (3-“factor” time-inhomogeneous Lévy process)

## The Lévy LIBOR model – construction

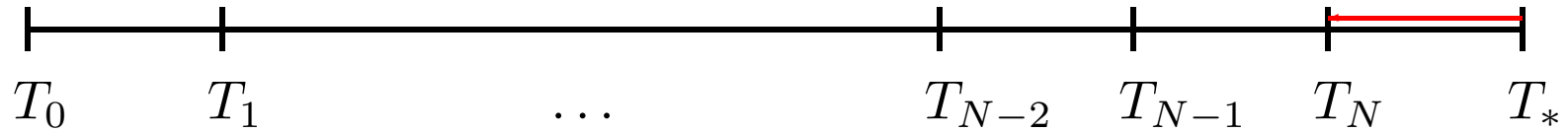
Discrete tenor structure:  $T_0 < T_1 < T_2 < \dots < T_N < T_{N+1} = T_*$ ,  $\delta_i = T_i - T_{i-1}$ .

**(LR1):** For any maturity  $T_i$  there exists a bounded, continuous, deterministic volatility function  $\lambda(\cdot, T_i) : [0, T_i] \rightarrow \mathbb{R}$ , corresponding to  $L(\cdot, T_i)$ . Moreover,

$$\sum_{i=1}^N |\lambda(s, T_i)| \leq M, \quad \forall s \in [0, T_*].$$

**(LR2):** Assume a strictly positive and strictly decreasing initial term structure  $B(0, T)$ ,  $T \in (0, T_*]$ . Consequently

$$L(0, T) = \frac{1}{\delta} \left( \frac{B(0, T)}{B(0, T + \delta)} - 1 \right) > 0.$$



Let  $\mathbb{P}_{T_*}$  be the terminal forward martingale measure. Postulate that the LIBOR with the longest maturity has dynamics

$$L(t, T_N) = L(0, T_N) \exp \left( \int_0^t b^L(s, T_N) ds + \int_0^t \lambda(s, T_N) dH_s \right), \quad (6)$$

where  $H$  is a  $\mathbb{P}_{T_*}$ -Lévy process, with canonical decomposition

$$H = \int_0^\cdot \sqrt{c_s} dW_s^T + \int_0^\cdot \int_{\mathbb{R}} x (\mu^H - \nu^{T_*})(ds, dx) \quad (7)$$

and the drift term is

$$b^L(s, T_N) = -\frac{1}{2} \lambda^2(s, T_N) c_s - \int_{\mathbb{R}} \left( e^{\lambda(s, T_N)x} - 1 - \lambda(s, T_N)x \right) F_s^{T_*}(dx). \quad (8)$$

Equivalently,

$$L(t, T_N) = L(0, T_N) \mathcal{E} \left( Z(T_N) \right)_t, \quad (9)$$

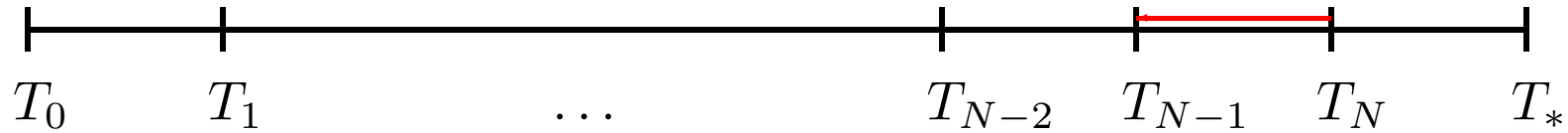
where  $Z(T_N)$  is the exponential transform of  $\int_0^\cdot \lambda(s, T_N) dH_s$ , i.e.

$$Z(T_N) = \int_0^\cdot \lambda(s, T_N) dH_s + \int_0^\cdot \int_{\mathbb{R}} (e^{\lambda(s, T_N)x} - 1 - \lambda(s, T_N)x) \mu^H(ds, dx). \quad (10)$$

Therefore,  $L(\cdot, T_N)$  satisfies the SDE

$$dL(t, T_N) = L(t-, T_N) dZ_t(T_N), \quad (11)$$

with initial condition  $L(0, T_1) = \frac{1}{\delta} \left( \frac{B(0, T_N)}{B(0, T)} - 1 \right)$ .



Define the forward measure  $\mathbb{P}_{T_N}$  via

$$\frac{d\mathbb{P}_{T_N}}{d\mathbb{P}_{T_*}} = \frac{F(T_N, T_N, T_*)}{F(0, T_N, T_*)} = \frac{1 + \delta L(T_N, T_N)}{1 + \delta L(0, T_N)}. \quad (12)$$

Then, the dynamics of the forward rate is

$$\begin{aligned} dF(t, T_N, T_*) &= \delta dL(t, T_N) \\ &= F(t-, T_N, T) \left( \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)} \lambda(t, T_N) dH_t \right. \\ &\quad \left. + \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)} \int_{\mathbb{R}} (e^{\lambda(t, T_N)x} - 1 - \lambda(s, T_N)x) \mu^H(dt, dx) \right). \end{aligned} \quad (13)$$

Now, the next LIBOR rate has dynamics

$$L(t, T_{N-1}) = L(0, T_{N-1}) \exp \left( \int_0^t b^L(s, T_{N-1}) ds + \int_0^t \lambda(s, T_{N-1}) dH_s^{T_N} \right), \quad (14)$$

where  $H^{T_N}$  is a **semimartingale** with canonical decomposition

$$H^{T_N} = \int_0^\cdot \sqrt{c_s} dW_s^{T_N} + \int_0^\cdot \int_{\mathbb{R}} x (\mu^H - \nu^{T_N})(ds, dx); \quad (15)$$

while  $b^L(\cdot, T_{N-1})$  ensures martingality. Here the  $\mathbb{P}_{T_N}$ -Brownian motion is

$$W^{T_N} = W^{T_*} - \int_0^\cdot \frac{\delta L(s-, T_N)}{1 + \delta L(s-, T_N)} \lambda(s, T_N) \sqrt{c_t} ds, \quad (16)$$

and the  $\mathbb{P}_{T_N}$ -compensator of  $\mu^H$  is

$$\nu^{T_N}(dt, dx) = \left( \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)} (e^{\lambda(t, T_N)x} - 1) + 1 \right) \cdot \nu^{T_*}(dt, dx). \quad (17)$$

The construction proceeds inductively; the dynamics of the LIBOR  $L(\cdot, T_i)$  under the forward measure  $\mathbb{P}_{T_{i+1}}$ , is

$$L(t, T_i) = L(0, T_i) \exp \left( \int_0^t b^L(s, T_i) ds + \int_0^t \lambda(s, T_i) dH_s^{T_{i+1}} \right), \quad (18)$$

where  $b^L(\cdot, T_i)$  ensures the martingality of the rate, and

$$H^{T_{i+1}} = \int_0^\cdot \sqrt{c_s} dW_s^{T_{i+1}} + \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(ds, dx). \quad (19)$$

$W^{T_{i+1}}$  is a  $\mathbb{P}_{T_{i+1}}$ -Brownian motion and the  $\mathbb{P}_{T_{i+1}}$ -compensator of  $\mu^H$  is

$$\nu^{T_{i+1}}(ds, dx) = \prod_{l=i+1}^N \underbrace{\frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} \left( \left( e^{\lambda(s, T_l)x} - 1 \right) + 1 \right)}_{:=\beta(s, x, T_l)} \nu^T(ds, dx). \quad (20)$$

► Consequence:  $H^{T_{i+1}}$  is **not** a time-inhomogeneous Lévy process (unless  $i = N$ ).

## Frozen compensator approximation

Idea: replace the random terms by their deterministic **initial values**

$$\frac{\delta L(t-, T_l)}{1 + \delta L(t-, T_l)} \approx \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)}$$

in the compensator  $\nu^{T_{i+1}}$  of  $\mu^H$  under the forward martingale measure  $\mathbb{P}_{T_{i+1}}$ ; i.e

$$\nu^{0, T_{i+1}}(ds, dx) = \prod_{l=i+1}^N \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} \left( \left( e^{\lambda(s, T_l)x} - 1 \right) + 1 \right) \nu^{T_*}(ds, dx). \quad (21)$$

- ▶  $H^{0, T_{i+1}}$  is a **time-inhomogeneous Lévy process**.
- ▶ Pricing is possible (but hard). Simulation is very hard ...
- ▶ What is the **error**? The error is compounded ...

## Terminal measure dynamics

We lift all LIBOR rates to the terminal measure  $\mathbb{P}_{T^*}$ :

$$L(t, T_i) = L(0, T_i) \exp \left( \int_0^t \bar{b}(s, T_i) ds + \int_0^t \lambda(s, T_i) dH_s \right), \quad (22)$$

where  $H$  is the *driving* time-inhomogeneous Lévy process, and

$$\begin{aligned} \bar{b}(s, T_i) = & -\frac{1}{2} \lambda^2(s, T_i) c_s - c_s \lambda(s, T_i) \sum_{l=i+1}^N \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l) \\ & - \int_{\mathbb{R}} \left( \left( e^{\lambda(s, T_i) x} - 1 \right) \prod_{l=i+1}^N \beta(s, x, T_l) - \lambda(s, T_i) x \right) F_s^{T^*}(dx). \end{aligned} \quad (23)$$

This can also be expressed as a [stochastic exponential](#).

Under the terminal measure, the dynamics of  $L(\cdot, T_i)$  satisfy the SDE

$$dL(t, T_i) = L(t-, T_i)dZ_t \quad (24)$$

where  $Z = (Z_t)_{0 \leq t \leq T}$  is a semimartingale with canonical decomposition

$$\begin{aligned} Z &= \int_0^\cdot b(s, T_i)ds + \int_0^\cdot \lambda(s, T_i)dH_s \\ &+ \int_0^\cdot \int_{\mathbb{R}} (e^{\lambda(s, T_i)x} - 1 - \lambda(s, T_i)x)\mu^H(ds, dx) \end{aligned} \quad (25)$$

and the (random) drift term is

$$\begin{aligned} b(s, T_i) &= -c_s \lambda(s, T_i) \sum_{l=i+1}^N \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l) \\ &- \int_{\mathbb{R}} \left( \left( e^{\lambda(s, T_i)x} - 1 \right) \prod_{l=i+1}^N \beta(s, x, T_l) - \lambda(s, T_i)x \right) F_s^{T^*}(dx). \end{aligned} \quad (26)$$

## Frozen drift approximation

Replace the random terms in the **drift** by their initial values, yielding:

$$b^0(s, T_i) = -c_s \lambda(s, T_i) \sum_{l=i+1}^N \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \lambda(s, T_l) - \int_{\mathbb{R}} \left( \left( e^{\lambda(s, T_i)x} - 1 \right) \prod_{l=i+1}^N \beta^0(s, x, T_l) - \lambda(s, T_i)x \right) F_s^{T^*}(dx). \quad (27)$$

where

$$\beta^0(t, x, T_l) = \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1. \quad (28)$$

- ▶ Each LIBOR is driven by a Lévy process. Simulation is *very easy*.
- ▶ Yields the same prices as the *frozen compensator* approximation.

# Strong Taylor approximation

**Aim:** develop a method to approximate the random terms, which allows to

1. incorporate and **improve** the “frozen drift/compensator” approximation,
2. **estimate** the error of the approximation;

↪ **strong Taylor approximation for perturbed SDEs.**

**Definition 1.** A *strong Taylor approximation* of order  $n \geq 0$  is a (truncated) power series

$$\mathbf{T}_\epsilon^n(W_\epsilon) := \sum_{i=0}^n \frac{\epsilon^i}{i!} \frac{\partial^i}{\partial \epsilon^i} \Big|_{\epsilon=0} W_\epsilon \quad (29)$$

such that

$$\mathbb{E} [ \|W_\epsilon - \mathbf{T}_\epsilon^n(W_\epsilon)\| ] = o(\epsilon^n), \quad (30)$$

holds true as  $\epsilon \rightarrow 0$ .

## Perturbed SDEs

Introduce a process  $X^\epsilon$ , parameterized by  $\epsilon$ , with dynamics

$$\begin{aligned} dX_t^\epsilon(T_i) = & \epsilon X_{t-}^\epsilon(T_i) \left( b^\epsilon(t, T_i) dt + \lambda(t, T_i) dH_t \right. \\ & \left. + \int_{\mathbb{R}} \left( e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x \right) \mu^H(dt, dx) \right), \end{aligned} \quad (31)$$

where

$$\begin{aligned} b^\epsilon(t, T_i) = & -c_t \lambda(t, T_i) \sum_{l=i+1}^N \frac{\delta_l X_{t-}^\epsilon(T_l)}{1 + \delta_l X_{t-}^\epsilon(T_l)} \lambda(t, T_l) \\ & - \int_{\mathbb{R}} \left( \left( e^{\lambda(t, T_i)x} - 1 \right) \prod_{l=i+1}^N \beta^\epsilon(t, x, T_l) - \lambda(t, T_i)x \right) F_t^{T*}(dx), \end{aligned} \quad (32)$$

and

$$\beta^\epsilon(t, x, T_l) = \frac{\delta_l X_{t-}^\epsilon(T_l)}{1 + \delta_l X_{t-}^\epsilon(T_l)} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1. \quad (33)$$

## First order Taylor approximation

The first order strong Taylor approximation of the random variable  $X_t^\epsilon(T_i)$  is :

$$\mathbf{T}_\epsilon^1(X_t^\epsilon(T_i)) = X_0^0(T_i) + \epsilon \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} X_t^\epsilon(T_i), \quad (34)$$

where  $X_0^0(T_i) \equiv X_t^0(T_i)$  for all  $t \in [0, T_i]$ . The first variation process of  $X^\epsilon(T_i)$ :

$$d \left( \underbrace{\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} X_t^\epsilon(T_i)}_{:= Y_t(T_i)} \right) = X_0^0(T_i) \left( b^0(t, T_i) dt + \lambda(t, T_i) dH_t + \int_{\mathbb{R}} (e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x) \mu^H(dt, dx) \right). \quad (35)$$

► Note that  $Y(T_i)$  is a Lévy process!

## Taylor approximation for the Lévy LIBOR model

Parameterize the LIBOR rate by  $\epsilon$ ; then approximate  $L^\epsilon(t, T_i)$  by

$$\begin{aligned} d\widehat{L}^\epsilon(t, T_i) = & \widehat{L}^\epsilon(t-, T_i) \left( \widehat{b}^\epsilon(t, T_i) dt + \lambda(t, T_i) dH_t \right. \\ & \left. + \int_{\mathbb{R}} (e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x) \mu^H(dt, dx) \right), \end{aligned} \quad (36)$$

where

$$\begin{aligned} \widehat{b}^\epsilon(t, T_i) = & -c_t \lambda(t, T_i) \sum_{l=i+1}^N \frac{\delta_l(\mathbf{T}_\epsilon^1(X_{t-}^\epsilon(T_l)))_+}{1 + \delta_l(\mathbf{T}_\epsilon^1(X_{t-}^\epsilon(T_l)))_+} \lambda(t, T_l) \\ & - \int_{\mathbb{R}} \left( (e^{\lambda(t, T_i)x} - 1) \prod_{l=i+1}^N \widehat{\beta}^\epsilon(t, x, T_l) - \lambda(t, T_i)x \right) F_t^{T^*}(dx), \end{aligned} \quad (37)$$

and

$$\widehat{\beta}^\epsilon(t, x, T_l) = \frac{\delta_l(\mathbf{T}_\epsilon^1(X_{t-}^\epsilon(T_l)))_+}{1 + \delta_l(\mathbf{T}_\epsilon^1(X_{t-}^\epsilon(T_l)))_+} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1. \quad (38)$$

## Error bounds

Using the Lipschitz property of the map  $x \mapsto \frac{\delta x}{1+\delta x}$ , we estimate

$$\begin{aligned}
 & \left| \log \widehat{L}^\epsilon(t, T_i) - \log L^\epsilon(t, T_i) \right| \\
 & \leq \int_0^t c_s |\lambda(s, T_i)| \delta^* \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - \mathbf{T}_\epsilon^1(X_{s-}^\epsilon(T_l))_+ \right| |\lambda(s, T_l)| ds \\
 & + \int_0^t \int_{\mathbb{R}} \left| e^{\lambda(s, T_i)x} - 1 \right| \delta^* \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - \mathbf{T}_\epsilon^1(X_{s-}^\epsilon(T_l))_+ \right| \left| e^{\lambda(s, T_l)x} - 1 \right| F_s^{T*}(dx) ds \\
 & \leq \int_0^t \mathcal{C}(s) \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - \mathbf{T}_\epsilon^1(X_{s-}^\epsilon(T_l))_+ \right| ds,
 \end{aligned}$$

which is the difference between the actual and approximated **drift** terms.

## Caplet pricing

The price of a caplet, using the relationship between the terminal and the forward measures, can be expressed as

$$\begin{aligned}\mathbb{C}(K, T_i) &= \delta_i B(0, T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}} [(L(T_i, T_i) - K)^+] \\ &= \delta_i B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} \left[ \frac{d\mathbb{P}_{T_{i+1}}}{d\mathbb{P}_{T_*}} \Big|_{\mathcal{F}_{T_i}} (L(T_i, T_i) - K)^+ \right] \\ &= \delta_i B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} \left[ \prod_{l=i+1}^N \left( 1 + \delta_l L(T_i, T_l) \right) (L(T_i, T_i) - K)^+ \right], \quad (39)\end{aligned}$$

where the dynamics of  $L(\cdot, T_l)$  are described by (22) and (23).

► Pricing by an Euler-Maruyama Monte Carlo scheme

## Example – numerical illustration

We consider a NIG Lévy process and constant volatilities  $\lambda_i$ ,  $1 \leq i \leq N$ . The caplet with maturity  $T_N$  is simply

$$\mathbb{C}(K, T_N) = \delta B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} [(L(T_N, T_N) - K)^+], \quad (40)$$

where the dynamics of the LIBOR process is

$$L(t, T_N) = L(0, T_N) \exp \left( \lambda_N H_t - \kappa(\lambda_N) t \right). \quad (41)$$

The caplet with maturity  $T_i$  is

$$\mathbb{C}(K, T_i) = \delta B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} \left[ \prod_{l=i+1}^N (1 + \delta L(T_i, T_l)) (L(T_i, T_i) - K)^+ \right], \quad (42)$$

where the **dynamics** of  $L(\cdot, T_i)$  are

$$L(t, T_i) = L(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i) t - \text{Pol}_i \left( \kappa(\lambda_l), \int_0^t \frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} ds \right) \right) \quad (43)$$

where  $\text{Pol}_i$  denotes certain polynomials, stemming from (23).

For the **strong Taylor approximation**, the dynamics are

$$\widehat{L}^\epsilon(t, T_i) = \widehat{L}^\epsilon(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i) t - \text{Pol}_i \left( \kappa(\lambda_l), \int_0^t \frac{\delta \mathbf{T}^1(X_{s-}^\epsilon(T_l))_+}{1 + \delta \mathbf{T}^1(X_{s-}^\epsilon(T_l))_+} ds \right) \right), \quad (44)$$

where

$$\mathbf{T}^1(X_t^\epsilon(T_l)) = X_0^0(T_l) + \epsilon Y_t(T_l), \quad (45)$$

and  $Y(T_l)$  is a Lévy process with decomposition

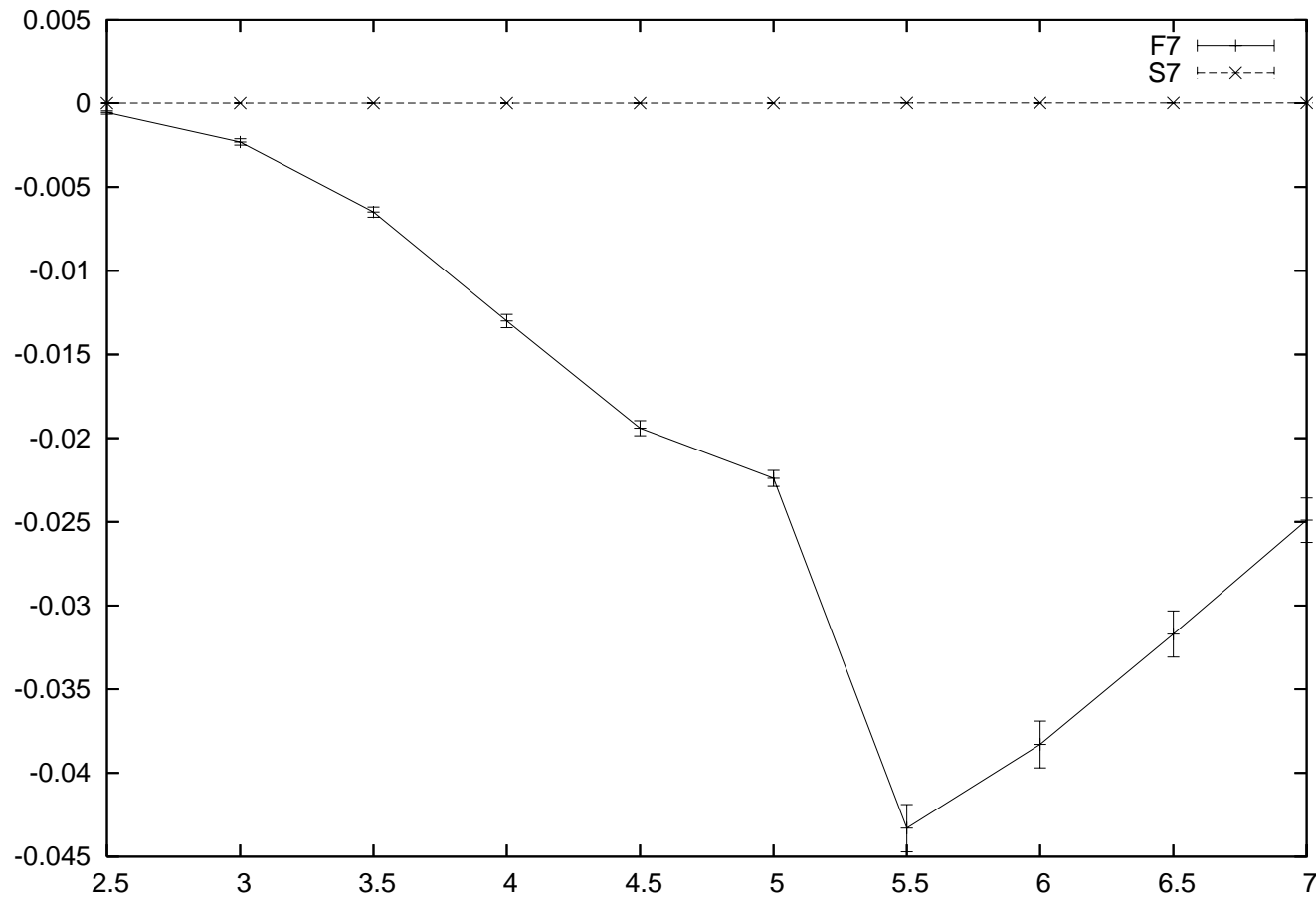
$$Y_t(T_l) = X_0^0(T_l) \left( \lambda_l H_t + \int_0^t \int_{\mathbb{R}} (e^{\lambda_l x} - 1 - \lambda_l x) \mu^H(ds, dx) - \text{Pol}_l \left( \kappa(\lambda_k) t, L(0, T_k) \right) \right). \quad (46)$$

Accordingly, for the **frozen drift approximation** the dynamics are

$$\widehat{L}^0(t, T_i) = \widehat{L}^0(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i) t - \text{Pol}_i \left( \kappa(\lambda_l), \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} t \right) \right). \quad (47)$$

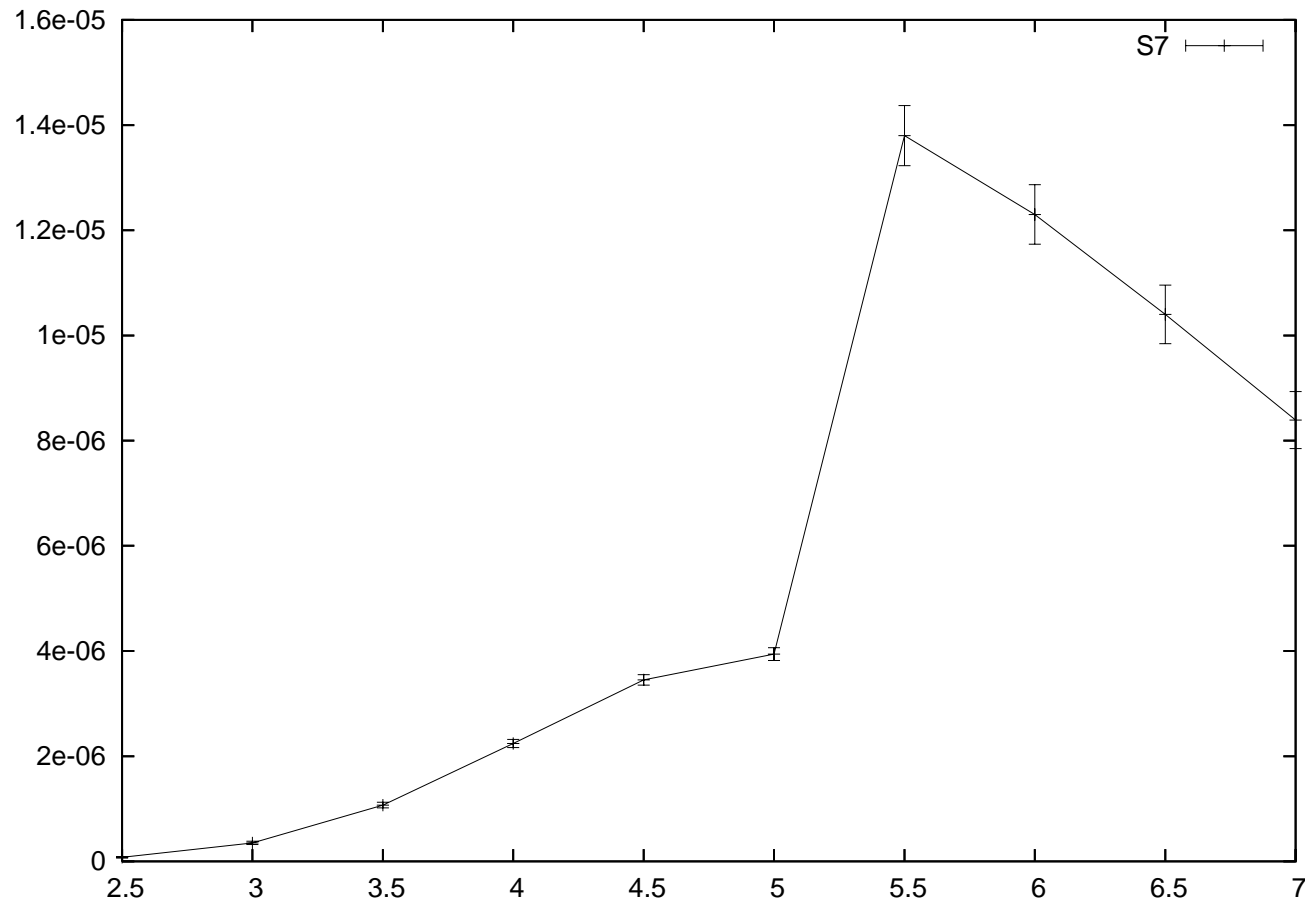
- ▶ Advantages: significant **reduction** in computational complexity
- ▶ Numerical examples: Siopacha & Teichmann (2007)  $\hookrightarrow$  diffusion models

# Numerical examples I



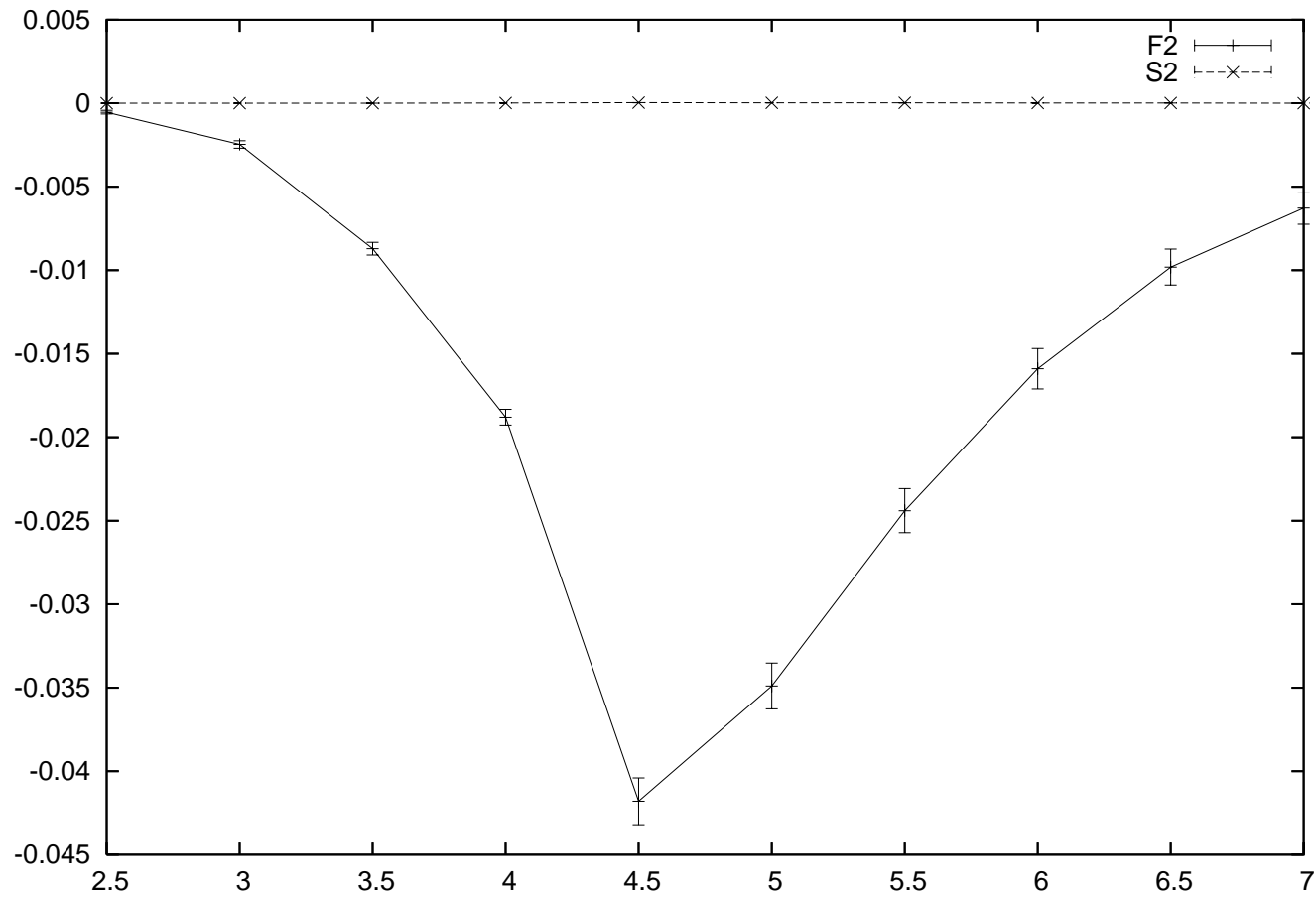
Error in bp. for caplet prices: full SDE – frozen drift approximation

## Numerical examples II



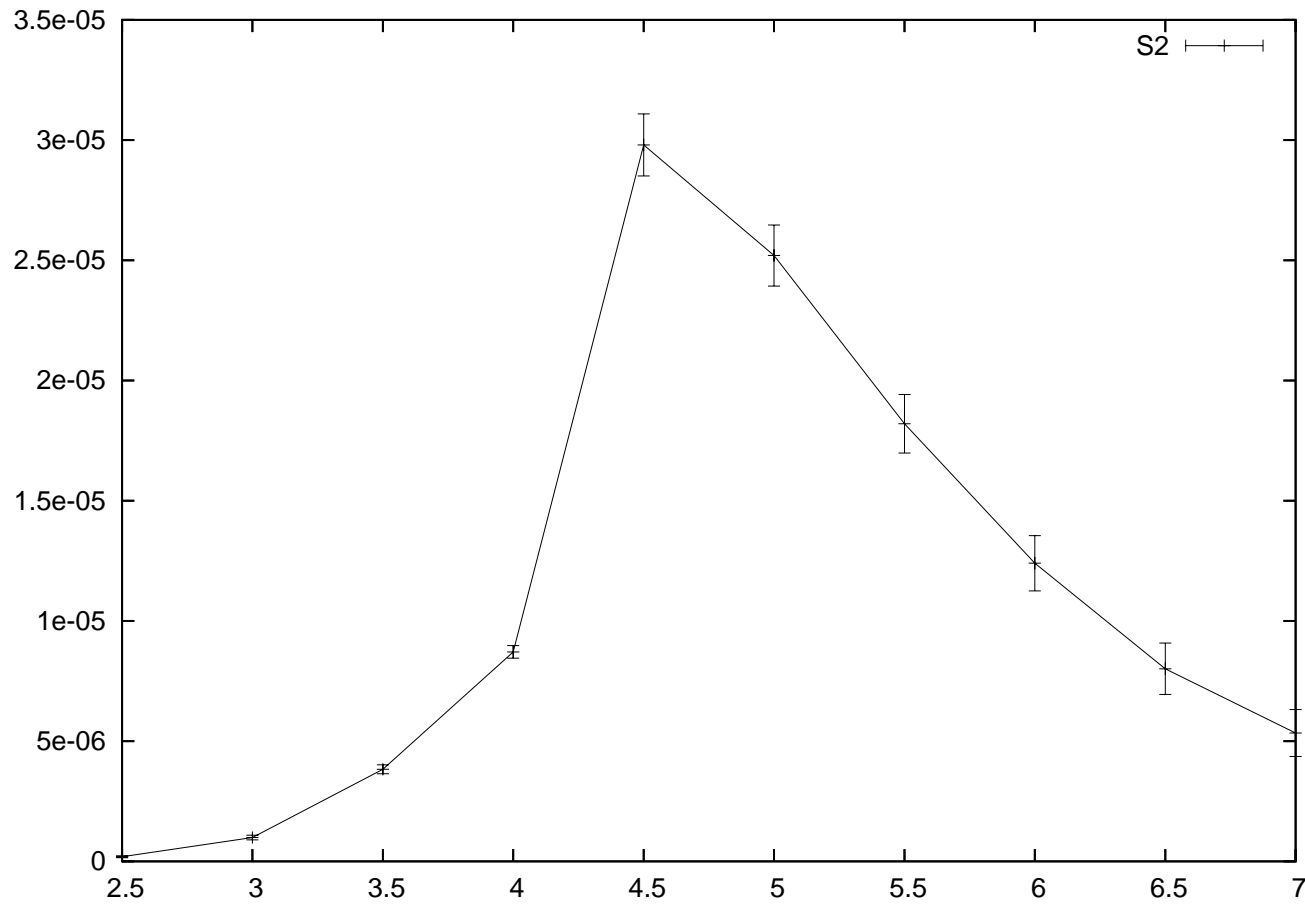
Error in bp. for caplet prices: full SDE – strong Taylor approximation

# Numerical examples III



Error in bp. for caplet prices: full SDE – frozen drift approximation

# Numerical examples IV



Error in bp. for caplet prices: full SDE – strong Taylor approximation

## Concluding remarks – Outlook

- ▶ The frozen drift approximation is quite accurate for [caplets](#)  $\rightsquigarrow$  fast calibration
- ▶ The strong Taylor approximation is useful for [complex payoffs](#)
- ▶ The method works for any semimartingale or Lévy-driven SDE ...
- ▶ Numerical illustrations (work in progress) ...
- ▶ Application in other settings: multi-currency, defaultable Lévy LIBOR model ...
- ▶ Preprint forthcoming on my webpage (google [papapantoleon](#)).

# References

- F. Black (1976). The pricing of commodity contracts. *J. Financ. Econ.* 3, 167-179.
- A. Brace, D. Gątarek, M. Musiela (1997). The market model of interest rate dynamics. *Math. Finance* 7, 127-155.
- D. Duffie, D. Filipovic, W. Schachermayer (2003). Affine processes and applications in finance. *Ann. Appl. Probab.* 13, 984-1053.
- E. Eberlein, F. Özkan (2005). The Lévy LIBOR model. *Finance Stoch.* 9, 327-348.
- J. Jacod, A.N. Shiryaev (2003). *Limit theorems for stochastic processes* (2nd. ed.). Springer.
- F. Jamshidian (1997). LIBOR and swap market models and measures. *Finance Stoch.* 1, 293-330.
- J. Kallsen, A. N. Shiryaev (2002). The cumulant process and Esscher's change of measure. *Finance Stoch.* 6, 397-428.
- W. Kluge (2005). *Time-inhomogeneous Lévy processes in interest rate and credit risk models*. Ph.D. thesis, University of Freiburg.
- K. R. Miltersen, K. Sandmann, D. Sondermann (1997). Closed form solutions for term structure derivatives with log-normal interest rates. *J. Finance* 52, 409-430.
- M. Musiela, M. Rutkowski (1997). Continuous-time term structure models: forward measure approach. *Finance Stoch.* 1, 261-291.
- K. Sandmann, D. Sondermann, K. R. Miltersen (1995). Closed form term structure derivatives in a Heath-Jarrow-Morton model with lognormal annually compounded interest rates. In *Proceedings of the Seventh Annual European Futures Research Symposium Bonn*, pp. 145-165.
- M. Siopacha, J. Teichmann (2007). Weak and strong Taylor methods for numerical solutions of stochastic differential equations. Preprint arXiv/0704.0745.