

Modeling and Estimation of Dependent Credit Rating Transitions

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Outline

- 1 Introduction
- 2 The Model
- 3 Simulation
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- Credit ratings describe the credit-worthiness of firms.
- We observe dependent changes of credit ratings of different firms.
- Without modeling the dependence between defaults we underestimate the risk.
- To model dependence, we apply interacting particle systems.

Advantage: Intuitive way to describe the dependence

Model assumption:

Credit ratings follow a time-homogeneous Markov jump process with the dynamics of an interacting particle system.

Applications in the literature:

- Giesecke and Weber (2004): Application of a voter model
- Bielecki and Vidozzi (2006), Frey and Backhaus (2007): Intensity of a credit rating transition for each firm, which depends on the configuration of the credit ratings
- Dai Pra, Runggaldier, Sartori and Tolotti (2007): Mean-field interaction model

Strongly Coupled Random Walk Process

Dynamics:

Independent Poisson processes with intensity $\lambda(x) \geq 0$
for each rating class x .

When Poisson process for rating x jumps, then:

- Rating class y is chosen with probability $P(x, y)$.
- Every firm with rating x tosses a coin with probability p_x of heads, independently of the other firms.
- If head occurs, then the firm changes the rating class from x to y .

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- State Space S^n
- State Space \mathcal{S}
- Preserving the Markov Property
- Embedding of a Model with Fewer Firms

3 Simulation

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Notation and State Spaces

Notation:

- Credit rating classes: $S = \{1, \dots, K\}$, where K means the firm is in default and 1 is the best rating class
- Firms: $F = \{1, \dots, n\}$

Possible state spaces:

- 1 Assigning a rating to each individual firm:
 - State space S^n

If the firms are indistinguishable, we use the state space:

- 2 Counting the number of firms in the rating classes:
 - State space: $\mathcal{S} = \{\eta \in \{0, \dots, n\}^S : \sum_{x \in S} \eta(x) = n\}$
 - $\eta \in \mathcal{S}$: $\eta(x)$ is the number of firms in rating class x

Parameters of the Credit Rating Process

Parameters:

- $\mu = (\mu_{xy})_{x,y \in S}$: Matrix of transition intensities (Q-matrix) of an individual firm
- $\rho = (\rho_x)_{x \in S} \in [0, 1]^S$: Dependence vector

If $\rho_x \in (0, 1]$, define:

- The jump intensity of the Poisson process:

$$\lambda(x) = \frac{\mu_x}{\rho_x}$$

$\mu_x = -\mu_{xx}$: Intensity of a single firm to leave rating x

- The probability of a rating change from x to y :

$$P(x, y) = \frac{\mu_{xy}}{\mu_x}, \quad x \neq y, \mu_x > 0$$

Process with State Space S^n

Process with state space S^n :

- $(X_t)_{t \geq 0}$: Markov jump process with state space S^n , describing the individual credit ratings of the n firms.
- X has the dynamics of the strongly coupled random walk.

Transition intensities:

- Independent Poisson processes at the rating classes
⇒ Intensity of a change of $k \geq 2$ firms with different ratings is zero.
- Intensity of a change of $k \geq 2$ firms to different ratings is zero.

Definition of Feasible Transitions of $(X_t)_{t \geq 0}$

- **Feasible transition:**

The credit rating of B firms changes from one rating class x to another rating class $y \neq x$, where A firms had originally rating x .

- Intensity of such a feasible transition:

$$\lambda(x)P(x, y)p_x^B(1 - p_x)^{A-B}$$

- Using the matrix μ of transition intensities of a single firm:

$$\mu_{xy}p_x^{B-1}(1 - p_x)^{A-B}$$

If $p_x = 0$, then the intensity of a change of exactly one firm is μ_{xy} and zero otherwise.

⇒ The firms with rating x move independently.

Q-Matrix of the Process with State Space S^n

Notation:

- $z, \tilde{z} \in S^n$: Rating configurations
- Transition $z \rightarrow \tilde{z}$ feasible:
 B firms change from rating class x to rating class $y \neq x$,
where A firms had originally rating x .
- A_u : Number of firms with rating u in z

Q-Matrix:

$$Q_n(z, \tilde{z}) = \begin{cases} \mu_{xy} p_x^{B-1} (1 - p_x)^{A-B}, & \text{if } z \rightarrow \tilde{z} \text{ is feasible,} \\ - \sum_{u \in S} \mu_u \sum_{j=0}^{A_u-1} (1 - p_x)^j, & \text{if } z = \tilde{z}, \\ 0 & \text{otherwise.} \end{cases}$$

Process with State Space \mathcal{S} (Indist. Firms)

- $(\eta_t)_{t \geq 0}$: Markov jump process with state space \mathcal{S} , which describes the number of firms in the rating classes
- η_{xy}^k : In configuration η , k firms with rating x change to $y \neq x$.
- Intensity of a change from η to η_{xy}^k :

$$Q^{\mathcal{S}}(\eta, \eta_{xy}^k) = \binom{\eta(x)}{k} \mu_{xy} p_x^{k-1} (1 - p_x)^{\eta(x)-k}$$

- Q -matrix of the process:

$$Q^{\mathcal{S}}(\eta, \eta') = \begin{cases} Q^{\mathcal{S}}(\eta, \eta_{xy}^k), & \text{if } \eta' = \eta_{xy}^k \text{ for } x, y \in \mathcal{S} \\ & \text{and } k \in \{1, \dots, \eta(x)\}, \\ - \sum_{\substack{x, y \in \mathcal{S} \\ x \neq y}} \sum_{k=1}^{\eta(x)} Q^{\mathcal{S}}(\eta, \eta_{xy}^k), & \text{if } \eta = \eta', \\ 0, & \text{otherwise.} \end{cases}$$

Preserving the Markov Property

Lemma (Function of MP again MP)

Assumptions:

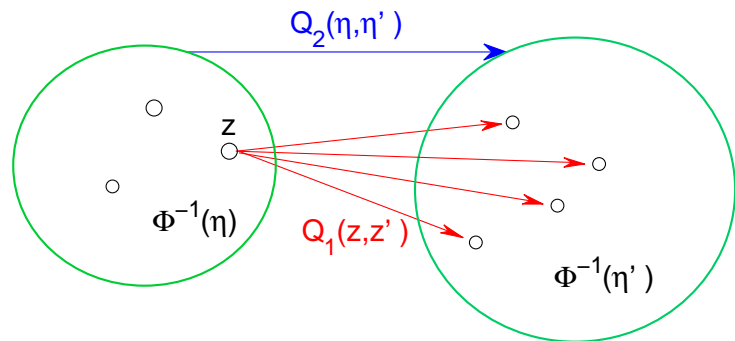
- S_1, S_2 : *Finite state spaces*
- $\Phi : S_1 \rightarrow S_2$: *Arbitrary function*
- $(X_t)_{t \geq 0}$: *Markov jump process w.r.t. the filtration $(\mathcal{F}_t)_{t \geq 0}$, generated by a Q-matrix Q_1 and state space S_1*
- Q_2 : *Q-matrix, such that for all $\eta, \eta' \in S_2$ with $\eta \neq \eta'$*

$$Q_2(\eta, \eta') = \sum_{z' \in \Phi^{-1}(\eta')} Q_1(z, z'), \quad \text{for all } z \in \Phi^{-1}(\eta)$$

$\Rightarrow (\Phi(X_t))_{t \geq 0}$ *is a Markov jump process w.r.t. $(\mathcal{F}_t)_{t \geq 0}$ with state space S_2 , generated by the Q-matrix Q_2 .*

Illustration of the Condition

$$Q_2(\eta, \eta') = \sum_{z' \in \Phi^{-1}(\eta')} Q_1(z, z'), \quad \text{for all } z \in \Phi^{-1}(\eta)$$



Strongly Coupled Random Walk is Dynamics of $(X_t)_{t \geq 0}$

Theorem

Assumptions:

- $(X_t)_{t \geq 0}$: Markov jump process with Q -matrix Q_n and state space S^n
- $\Phi : S^n \rightarrow S$, where for $z = (z_1, \dots, z_n) \in S^n$

$$(\Phi z)(x) = \sum_{i=1}^n \mathbb{1}_{\{z_i=x\}}, \quad x \in S$$

$\Rightarrow \eta_t = \Phi(X_t)$ for $t \geq 0$ is a Markov jump process with state space S and Q -matrix Q^S .

Embedding of a Model with Fewer Firms

Theorem

Assumptions:

- $m, n \in \mathbb{N}$ with $m < n$: Number of firms
- $(X_t)_{t \geq 0}$: Markov jump process with state space S^n and Q -matrix Q_n
- $\pi : S^n \rightarrow S^m$: Projection with $\pi(x) = x|_{S^m}$

$\Rightarrow Y_t = \pi(X_t)$ for $t \geq 0$ is a Markov jump process with state space S^m , generated by Q -matrix Q_m .

Intensity of a rating change of m firms in a model with n firms
= Intensity of a change in a model with m firms

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 - Portfolio
 - Simulation of the Profit and Loss
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Portfolio and Profit-and-Loss Function

Portfolio:

- n defaultable zero-coupon bonds issued by n firms with face value f_i and maturity T_i for $i = 1, \dots, n$
- Simplification: No recovery and zero default-free interest rate

Portfolio value at future time $t \leq \min\{T_1, \dots, T_n\}$:

$$\begin{aligned} V(t) &= \sum_{i=1}^n f_i \mathbf{P}[X_i(T_i) \neq K | X_t] \\ &= \sum_{i=1}^n f_i \left(1 - \left(\exp\{\mu(T_i - t)\} \right)_{X_i(t), K} \right) \end{aligned}$$

Profit-and-loss during $(0, t]$:

$$\text{P\&L}(t) = V(t) - V(0)$$

Simulation of the Loss of the Portfolio

Model assumptions:

- $K = 8$ credit rating classes
- Initial rating: All firms have rating 4.
- Movement according to our model with generator μ based on data of Standard & Poor's
- Dependence vector $(p_x)_{x \in S}$: $p_x = p$ for all $x \in S$

Portfolio assumptions:

- Number of bonds: $n = 100$
- Face value: $f_i = 1$ for every bond
- Common maturity: $T_i = 10$ years

Distribution Function of Profit and Loss

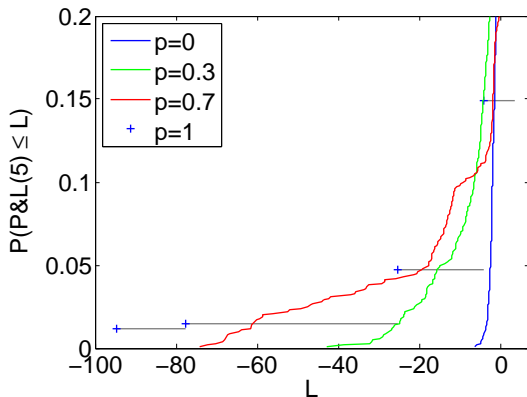


Figure: Distribution of the profit and loss of the portfolio at time $t = 5$ for $n = 100$ firms (1 000 simulations). The dependence parameter p_x equals p for each rating class. Initial value of the portfolio is $V(0) = 95$.

Distribution Function of Profit and Loss

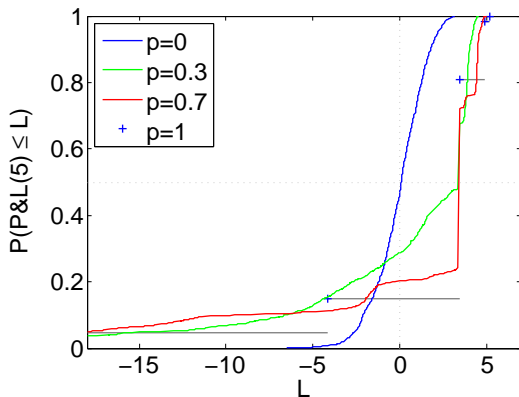


Figure: Distribution of the profit and loss of the portfolio at time $t = 5$ for $n = 100$ firms (1 000 simulations). The dependence parameter p_x equals p for each rating class. Initial value of the portfolio is $V(0) = 95$.

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 - Likelihood Function
 - Maximum Likelihood Estimator
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Rating Process with Varying Number of Firms

Notation:

- $(X_{t,n})_{t \geq 0}$ for $n \in \mathbb{N}$: Markov jump process with state space S^n and Q -matrix Q_n
- $X_{0,n}$: Initial rating, independent of the parameters ρ and μ
- N : Random number of firms, independent of $(X_{t,n})_{t \geq 0}$ for all $n \in \mathbb{N}$ and independent of the parameters ρ and μ

Credit rating process with N firms:

$$(Y_t)_{t \geq 0} = (X_{t,N})_{t \geq 0}$$

We observe independent samples of the credit rating process $(Y_t)_{t \in [0, T)}$.

Space of the Sample Paths of $X_{t,n}$

Take modification of Y with piecewise constant sample paths with finite number of jumps

⇒ Sample path of the process $(X_{t,n})_{t \in [0, T]}$, which jumps m times in $[0, T)$:

$$((z_0, t_0), \dots, (z_{m-1}, t_{m-1}), z_m)$$

That means:

- Process starts in z_0 and remains there for time t_0 .
- Then it jumps to z_1 and remains there for time t_1 .
- \vdots
- Finally it reaches z_m and stays there until T .

Space of the Sample Paths of Y

Space of the sample paths of $(X_{t,n})_{t \in [0, T]}$ with m jumps:

$$\mathcal{S}_{n,m} = (S^n \times \mathbb{R})^m \times S^n \quad \text{for } m \in \mathbb{N}$$

$$\mathcal{S}_{n,0} = S^n \quad \text{for } m = 0$$

Space of sample paths of $(X_{t,n})_{t \in [0, T]}$:

$$\mathcal{S}_n = \bigcup_{m=0}^{\infty} \mathcal{S}_{n,m}$$

Space of sample paths of the process $Y = X_N$:

$$\mathcal{S}^Y = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

Measure on Space of Sample Paths

- \mathcal{V} : Lebesgue–Borel measure on \mathbb{R}
- \mathcal{N}_n : Counting measure on S^n
- $\sigma_{n,m}$: Product measure on $S_{n,m}$ with

$$\sigma_{n,m} = (\mathcal{N}_n \otimes \mathcal{V})^{\otimes m} \otimes \mathcal{N}_n$$

- \mathcal{B} : Smallest σ -algebra containing all sets of S^Y whose intersection with $S_{n,l}$ is a Borel set for each pair $(n, l) \in \mathbb{N} \times \mathbb{N}_0$

Define the measures:

$$\sigma_n(C \cap S_n) = \sum_{m=0}^{\infty} \sigma_{n,m}(C \cap S_{n,m}) \quad \text{for } C \in \mathcal{B}$$

and

$$\sigma(C) = \sum_{n=1}^{\infty} \sigma_n(C \cap S_n) \quad \text{for } C \in \mathcal{B}$$

Feasible Path of $X_{t,n}$

Feasible path of $(X_{t,n})_{t \in [0, T]}$:

- Start in z_0 is possible, i.e. $\mathbf{P}(X_{0,n} = z_0) > 0$.
- All transitions are feasible.

Define for feasible path ω :

- $N_{x,y,A,B}^{(\omega)}$: Number of simultaneous transitions of B firms from rating class x to $y \neq x$, A firms originally with rating x
- $T_{x,A}^{(\omega)}$: Total time that exactly A firms have rating x

Density of a Sample Path

Theorem

Let E be a measurable subset of \mathcal{S}^Y . Then for $n \in \mathbb{N}$ with $\mathbf{P}(N = n) > 0$

$$\mathbf{P}((Y_t)_{t \in [0, T]} \in E \mid N = n) = \int_{E \cap \mathcal{S}_n} f_{Q_n}(\omega) \sigma_n(d\omega),$$

where $f_{Q_n}(\omega) = 0$, if the path is not feasible,
and for a feasible path $\omega \in \mathcal{S}_n$

$$f_{Q_n}(\omega) = \mathbf{P}(X_{0,n} = z_0) \exp \left\{ - \sum_{A=1}^n \sum_{x \in \mathcal{S}} T_{x,A}^{(\omega)} \mu_x \sum_{j=0}^{A-1} (1 - p_x)^j \right\} \\ \times \prod_{\substack{x,y \in \mathcal{S} \\ x \neq y}} \prod_{\substack{A,B=1 \\ A \geq B}}^n \left(\mu_{xy} p_x^{B-1} (1 - p_x)^{A-B} \right)^{N_{x,y,A,B}^{(\omega)}}.$$

Likelihood Function

- $\omega_1, \dots, \omega_k \in \mathcal{S}^Y$: Feasible sample paths
- n_j : Number of firms in path ω_j
- $n := \max_{j \in \{1, \dots, k\}} n_j$: Maximal number of observed firms
- $N_{x,y,A,B}$: Total number of simultaneous rating changes of B firms from x to $y \neq x$, A firms originally with rating x
- $T_{x,A}$: Total time that exactly A firms have rating x

Conditional likelihood function, given $N_1 = n_1, \dots, N_k = n_k$:

$$\mathcal{L}(\omega_1, \dots, \omega_k) = \prod_{\substack{x,y \in \mathcal{S} \\ x \neq y}} \prod_{\substack{A,B=1 \\ A \geq B}}^n \left(\mu_{xy} p_x^{B-1} (1 - p_x)^{A-B} \right)^{N_{x,y,A,B}} \\ \times \left(\prod_{j=1}^k \mathbf{P}(X_{0,n_j} = z_0^j) \right) \exp \left\{ - \sum_{x \in \mathcal{S}} \mu_x \sum_{A=1}^n T_{x,A} \sum_{j=0}^{A-1} (1 - p_x)^j \right\}$$

Credit Rating Sets

Subsets of the rating set $S = \{1, \dots, K\}$, depending on the observed feasible paths of Y :

- $x \in S^{T>0}$, iff at least one firm has rating x temporarily during the observation period
⇒ Parameter estimation is possible
- $x \in S^{p>0}$, iff at least two firms with rating x change the rating simultaneously at some time
⇒ Firms change the rating x not independently
- $x \in S^{p<1}$, iff at least at one time not all firms with rating x change this rating together
⇒ Firms are not totally dependent

Maximum Likelihood Estimator for the Q-Matrix μ

Theorem (MLE for μ)

A maximum likelihood estimator (MLE) of the parameter vector $[p, \mu]$ is given as follows.

(a) For $(x, y) \in S^{T>0} \times S$ with $x \neq y$ the MLE of entry μ_{xy} :

$$\hat{\mu}_{xy} = \frac{\sum_{A=1}^n \sum_{B=1}^A N_{x,y,A,B}}{\sum_{A=1}^n T_{x,A} \sum_{j=0}^{A-1} (1 - \hat{p}_x)^j},$$

where \hat{p}_x is the MLE of p_x , which is given in the next slides.

\Rightarrow If $p_x = 0$, then we get the same MLE in this model for μ as in the independent model:

$$\hat{\mu}_{xy}^{\text{ind}} = \frac{\sum_{A=1}^n N_{x,y,A,1}}{\sum_{A=1}^n A T_{x,A}}$$

Max. Likelihood Estimator for Dep. Parameter p_x

Theorem (MLE for p)

(b) For the rating classes $x \in S^{p>0} \cap S^{p<1}$ the MLE of p_x is uniquely determined. The MLE is the unique root in $(0, 1)$ of the polynomial of degree $\leq n$ with coefficients

$$c_0 = \sum_{\substack{A,B=1 \\ A \geq B}}^n (B-1) \tilde{N}_{x,A,B} \sum_{i=1}^n i T_{x,i}$$

$$c_j = (-1)^j \sum_{\substack{A,B=1 \\ A \geq B}}^n \tilde{N}_{x,A,B} \sum_{i=j}^n \binom{i}{j} \left(\frac{i-j}{j+1} B + A - i \right) T_{x,i}$$

for $j \in \{1, \dots, n\}$.

$\tilde{N}_{x,A,B}$: Total number of rating changes of B firms with rating x , A firms originally with rating x

Max. Likelihood Estimator for Dep. Parameter p_x

Theorem (MLE for p)

(c) For $x \in S^{p>0} \setminus S^{p<1}$ the MLE is $\hat{p}_x = 1$.

In the remaining case of insufficient observations:

(d) If $x \in S^{p<1} \setminus S^{p>0}$, the MLE for p_x is one of the following:

- $\hat{p}_x = 0$ or
- the unique root of the polynomial with coefficients as above.

(e) For $x \in S^{T>0} \setminus (S^{p>0} \cup S^{p<1})$ the MLE is $\hat{p}_x = 1$, if

- there is a change, where only one firm has rating x and is leaving x , and
- at least two firms have rating x for a time interval.

Further Research

Asymptotic properties of the estimator:

- Consistency?
- Asymptotical normality?

Extension of the model:

- Observations of the credit ratings only at discrete time points
- ⇒ Estimation of the parameters using historical rating transitions
- Independent copies of the process, each one symbolizing an sector of industry

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