FLEXIBLE COMPLETE MODELS WITH STOCHASTIC VOLATILITY GENERALISING HOBSON-ROGERS

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Abstract. We apply methods from Malliavin Calculus in the spirit of Fourny et al. in order to calculate Taylor expansion of model prices with respect to perturbation parameters. The methods allow to calculate certain random variables, so called weights, explicitly, which serve to prove analytically tractable formulas approximating actual model prices. We focus on hypo-elliptic rather than elliptic equations such as the Hobson-Rogers model.

This approach is then applied to investigate whether complete stochastic volatility models like the Hobson-Rogers model can produce appropriate smiles or not. Furthermore we suggest generalizations of the Hobson-Rogers model which remain complete, but fit the features of actual market data much better.

1. Introduction and Results

Most common stochastic volatility models for asset prices, such as [15, 11, 23], for example, include a non-traded source of risk, see also [17, 13, 22, 28, 19]. Thus, the corresponding markets are incomplete and option prices are not uniquely determined by no arbitrage arguments. An alternative approach models volatility as a deterministic function of the underlying, see for example [19]. In such models instantaneous volatility is perfectly correlated with the underlying price process. Recent empirical evidence in [25] for stock index options and [25], [27] for bond options and foreign exchange options does not support this hypothesis.

To overcome the deficiencies of deterministic stochastic volatility models Hobson and Rogers [12] introduce a class of complete stochastic volatility models, where the volatility is a function of the past (logarithms of) underlying prices. Related work is [30], and, slightly different in spirit, [31].

A concrete example from Hobson and Rogers [12] will be the starting point for the empirical part of this research. For brevity we call this model simply *the* Hobson-Rogers model (HR). In their paper the authors observe that the model effectively accounts for the possibilities of smiles, and illustrate this with several graphs, produced by numerically solving the corresponding partial differential equation. Recent work on the numerics of the model and variants is di Francesco et al. (see for instance [8]).

Despite of its attractive features, there is little echo in empirical research. We found only [29], where the ideas are extended to term-structure modeling.

In this paper we calibrate the Hobson-Rogers model to actual option prices using an analytical approximation, see below. We compare volatility surfaces produced by that model to actual volatility surfaces.

To this end, we considered options on British Pound/US Dollar futures, where we had settlement prices for the market for all traded options from 1996 to 2002. Using this data, we selected a number of (typical) days and estimated the implied volatility surface by numerically inverting the Barone-Adesi Whaley model (as these

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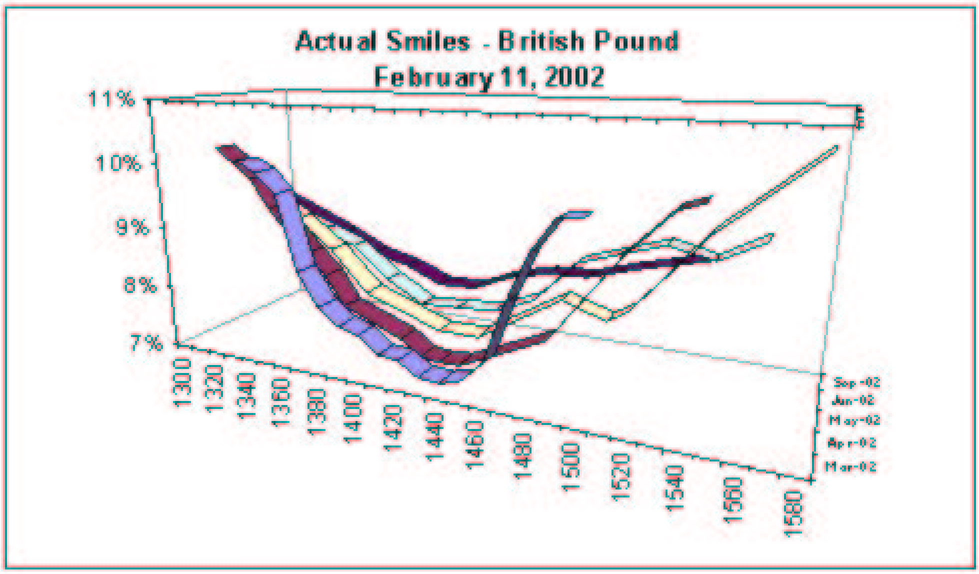

Figure 1

options are American style options on futures, see [2]) and then expressing the result in terms of the Black implied volatilities (European style options on futures, see [3]). The underlying asset price was the settlement price of the futures contract at the same moment as the settlement of all the options contracts and the interest rate used was the US Dollar Libor interpolated for the expiration of the options contract. As representative, we selected one day, 11 February 2002, and found five maturities were available, March 2002, April 2002, May 2002, June 2002 and September 2002. For each of these maturities, we had quoted call and put options in a fairly wide range and we determined the Black implied volatilities and plotted these across striking price and time to expiration (see Figure 1). We confirm, that the Hobson and Rogers model will produce what appears to be smiles and term structure effects for implied volatilities qualitatively (see Figure 2a). However, when the scale is chosen to match that of observed option implied volatility surfaces, the Hobson and Rogers surfaces appear to be almost flat (see Figure 2b). While this is just one particular instance, it seems quite intuitive, that the volatility process in the Hobson-Rogers model, being of moving average type, in general is not sufficiently wild to produce volatility smiles of the order of magnitude corresponding to the observed option prices, unless we allow unreasonable values of the parameters, that defy the original idea that the model is a correction to Black-Scholes. We claim that a simple generalisation of the Hobson and Rogers model, referred to as the generalised Hobson Rogers model (GHR) can do the job (see Figure 3).

In mathematical finance we model price behaviour by semi-martingales $(S_t^{(\epsilon)})_{t\geq 0}$, which often depend on additional parameters, here denoted by $\epsilon \geq 0$. We propose a method (inspired by the successful approach [10]), which allows to calculate derivatives of the function $\epsilon \mapsto E(\phi(S_T^{(\epsilon)}))$ efficiently. To be more precise, we are able to prove that – under some technical assumptions – there exist random variables $\pi^{(n)}$ such that

$$\frac{\partial^n}{\partial \epsilon^n} E(\phi(S_T^{(\epsilon)})) = E(\phi(S_T^{(\epsilon)})\pi^n).$$

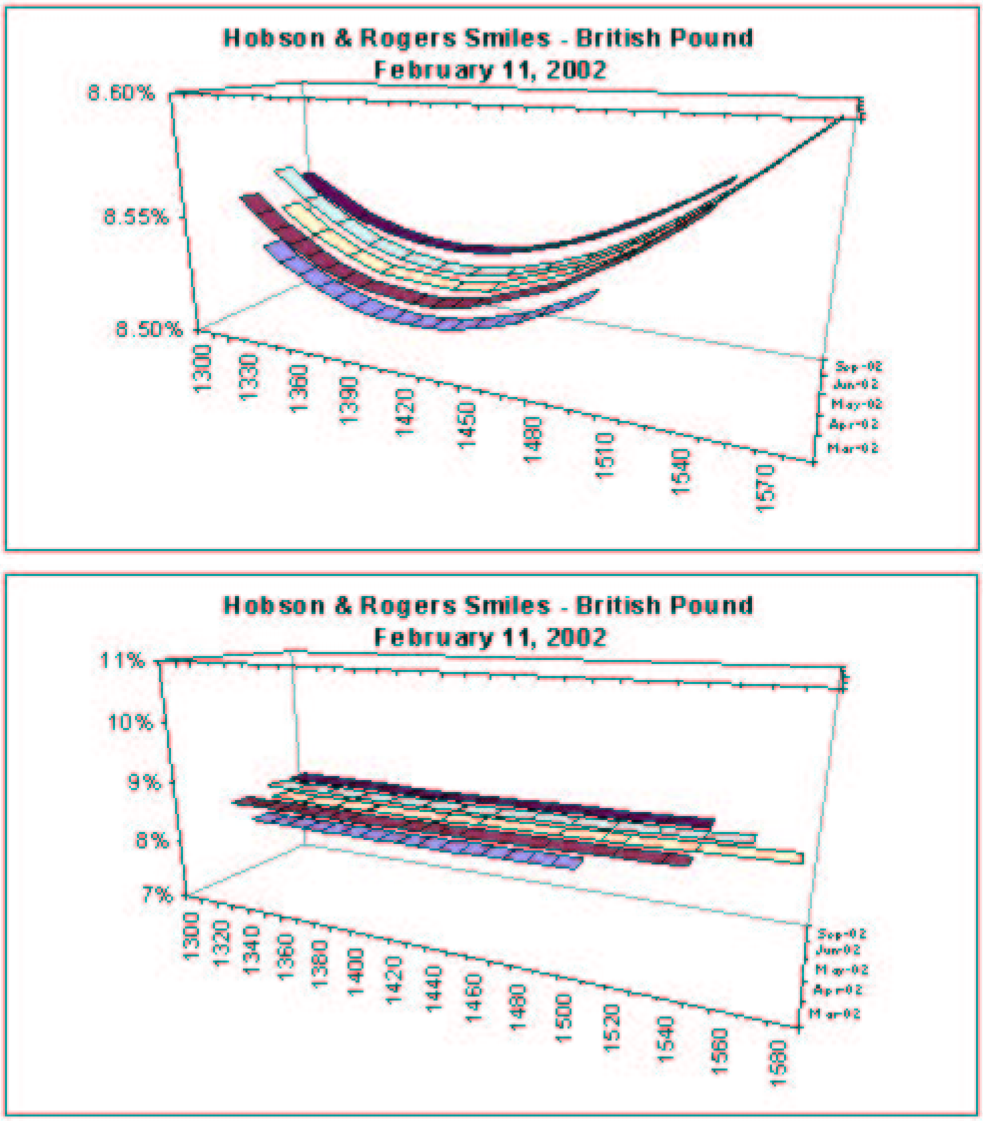

Figure 2

These results and particular methods how to calculate the weights π^n are proved in Theorem 1 and Theorem 2. In contrast to [10] we are interested in *hypo-elliptic* equations rather than elliptic ones, as for instance the Hobson-Rogers model. Additionally to the well-known PDE methods for expansions with respect to parameters (private communication with Sam Howison), the Malliavin Calculus approach provides

- explicit algorithms how to calculate the weights, even if the functions in question are not real analytic.
- probabilistic approaches for approximations of model prices, which allow efficient Monte-Carlo evaluations.

Hence we are able to calculate Taylor expansions of model prices to obtain analytically tractable approximations of the actual model prices. These analytically

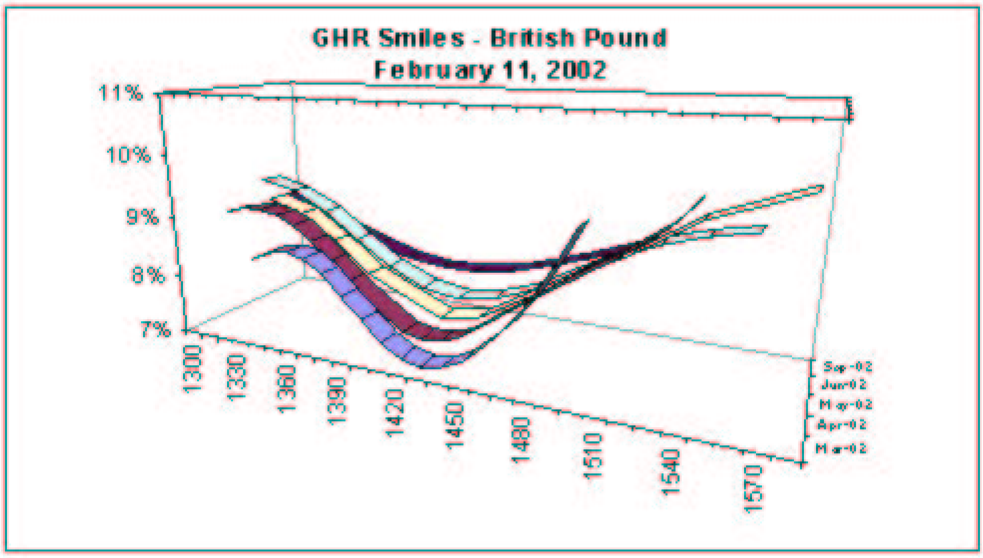

Figure 3

tractable approximations are then applied to the determination of model parameters, which yields an efficient method how to determine approximatively if a model works or not. The demonstration of this approach is exemplified in the previous Hobson-Rogers models.

The article is structured as follows: in Section 2 we introduce the main tools from 1-dimensional Malliavin Calculus and prove the main theorems. Furthermore we prove that the weights π^n can be calculated as polynomials of integrated Gaussian polynomials (see Remark 1). Two examples are added which correspond precisely to the model proposed by [12] and our generalisation of it. In Section 3 we introduce the models in question and calculate the respective approximative prices applying Theorem 1.

2. Partial Integration and Taylor expansion of Prices

We shall analyse models for the price returns, which are given as parameter-dependent stochastic processes $(X_t^{x,\epsilon})_{0\leq t\leq T}$ for a parameter $\epsilon \geq 0$: for $\epsilon = 0$ we find ourselves in a well-known model (for instance the Black-Scholes model). We are interested in Taylor expansions of prices $E(\phi(X_T^{x,\epsilon}))$ with respect to the parameter ϵ (the risk neutral measure does not depend on ϵ!). Therefore we introduce a convenient class of families $(F_\epsilon)_{\epsilon\geq 0}$ of random variables, whereon we can apply partial integration techniques. For more general considerations and additional results, see [24].

We shall apply the following techniques derived from Malliavin Calculus (see [18] and [21]), which are inspired by [10]. The method allows to write approximations of model prices by ordinary integration of polynomials with respect to a Gaussian density along simple subsets. We present the method in the simplest setting, i.e. a Gaussian probability space generated by a one dimensional Brownian motion.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, which is generated by a one-dimensional Brownian motion $(B_t)_{0\leq t\leq T}$ for some $T > 0$, i.e. $\mathcal{F} = \mathcal{F}_T$. For the reader who is familiar with Ito-integration, but does not feel comfortable with Malliavin Calculus, we list the following simple rules, which allow to follow all calculations which are done in the article:

(1) The Malliavin derivative associates to random variables (in its domain of definition) $X \in \mathrm{dom}(D) \subset L^2(\Omega)$ a not necessarily adapted process $(D_s X)_{0 \le s \le T} \in L^2([0,T] \times \Omega)$. The Malliavin derivative is a closed, densely defined, unbounded linear operator and the following rules hold,

$$D_s(1_\Omega) = 0 \tag{2.1}$$

$$D_s(\int_0^T \sigma(s) dB_s) = \sigma(s) 1_{[0,T]}(s), \tag{2.2}$$

$$D_s(\phi(X_1, \dots, X_n)) = \sum_{i=1}^n \frac{\partial}{\partial x^i} \phi(X_1, \dots, X_n) D_s X_i \tag{2.3}$$

for $X_i \in \mathrm{dom}(D)$, $i = 1, \dots, n$ and σ a square-integrable, deterministic function on $[0,T]$. ϕ is given as a C^1-function on $\mathbb{R}^n$.

(2) The adjoint of the Malliavin derivative is the Skorohod integral δ, which associates to a not necessarily adapted process $(Y_s)_{0 \le s \le T} \in \mathrm{dom}(\delta) \subset L^2([0,T] \times \Omega)$ a random variable $\delta(s \mapsto Y_s) \in L^2(\Omega)$. The Skorohod integral is a closed, densely defined, unbounded linear operator and the following basic partial integration formula holds true

$$E(X \delta(s \mapsto Y_s)) = E(\int_0^T (D_s X) Y_s ds) \tag{2.4}$$

on the respective domains. The most important, non-trivial assertion on Skorohod integration is the relation to Ito-integration: namely, for all square-inegrable, predictable processes $(Y_s)_{0 \le s \le T}$ we obtain that $(Y_s)_{0 \le s \le T} \in \mathrm{dom}(\delta)$ and

$$\delta(s \mapsto Y_s) = \int_0^T Y_s dB_s. \tag{2.5}$$

(3) By extension of the derivative operator D on L^p-spaces we obtain domains of definition $\mathcal{D}^{p,1} \subset L^p(\Omega)$. By definition of iterated derivatives on the respective domains we obtain domains of definition $\mathcal{D}^{p,n} \subset L^p(\Omega)$, where the Malliavin-derivative can be applied n times. Smooth random variables are those which lie in the domain of each derivative operator in each L^p, i.e.

$$\mathcal{D}^\infty = \cap_{p \ge 1} \cap_{n \ge 0} \mathcal{D}^{p,n}. \tag{2.6}$$

A fortiori smooth random variables are closed under composition with smooth, polynomially bounded functions and allow Skorohod integration up to arbitrary orders (see [18]).

(4) For Skorohod integrable process $(u_s)_{0 \le s \le T}$ and $F \in \mathcal{D}^\infty$ with $E(\int_0^T F^2 u_s^2 ds) < \infty$, the process $(F u_s)_{0 \le s \le T}$ is Skorohod-integrable and

$$\delta(s \mapsto u_s F) = F \delta(s \mapsto u_s) - \int_0^T u_s D_s F ds \tag{2.7}$$

holds true.

(5) The Malliavin covariance matrix is a real-valued random variable in the one-dimensional case,

$$\gamma(X) := \int_0^T (D_s X)^2 ds. \tag{2.8}$$

If $\gamma(X)$ is invertible almost surely, then X has a density with respect to Lebesgue's measure.

We shall deal with families of random variables $\epsilon \mapsto G_\epsilon$ such that

- for all $\epsilon \geq 0$ the random variable and all its derivatives with respect to ϵ are smooth, i.e. $\frac{\partial^k}{\partial \epsilon^k} G_\epsilon \in \cap_{p \geq 1} \cap_{n \geq 0} \mathcal{D}^{p,n}$ for $k \geq 0$, together with all Malliavin derivatives. The derivatives are taken with respect to the topology of $\mathcal{D}^\infty$, which is equivalent to the assertion that the maps $\epsilon \mapsto \eta(G_\epsilon)$ are smooth, for all continuous linear functionals $\eta : \mathcal{D}^\infty \to \mathbb{R}$.

We denote this space by $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$. Notice in particular that this space is a (smooth) algebra of random variables (see [18], Ch. II, 5.8), where the Skorohod integral and the Malliavin derivative are well-defined (see [18]). In particular the constant curve $\epsilon \mapsto 1$ satisfies the requirements. Observe the following rules of differentiation:

- Malliavin derivatives and Skorohod integrals commute with derivatives with respect to ϵ.
- all Malliavin derivatives of derivatives with respect to ϵ are Skorohod integrable.

Definition 1. *A family $(\epsilon \mapsto F_\epsilon) \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$, which additionally satisfies that the (Malliavin) covariance matrix $\gamma(F_\epsilon)$ is almost surely invertible for $\epsilon \geq 0$ and at $\epsilon = 0$ (but not necessarily off 0)*

$$\frac{1}{\gamma(F_0)} \in \mathcal{D}^\infty = \cap_{p \geq 1} \cap_{n \geq 0} \mathcal{D}^{p,n}, \tag{2.9}$$

is called a family with regular density*.*

Example 1. *We shall provide the following characteristic (and useful!) example for families with regular density: given a continuous Gaussian process $(S_t)_{t \geq 0}$ with*

$$dS_t = (a(t) - \lambda S_t)dt + b(t)dB_t \tag{2.10}$$

with continuous (deterministic), square integrable functions $a, b : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we define the family $(F_\epsilon)_{\epsilon \geq 0}$ for a fixed number $T > 0$ and show by simple calculations that $(F_\epsilon)_{\epsilon \geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$,

$$F_\epsilon := z - \frac{\eta^2}{2} \int_0^T (1 + \epsilon S_t^2)^2 dt + \eta \int_0^T (1 + \epsilon S_t^2) dB_t, \tag{2.11}$$

$$\begin{aligned} D_s F_\epsilon = \eta(1 + \epsilon S_s^2) - 2\epsilon\eta^2 \int_s^T (1 + \epsilon S_t^2) S_t \exp(-\lambda(T - t)) b(t) dt + \\ + 2\epsilon\eta \int_s^T S_t \exp(-\lambda(T - t)) b(t) dB_t, \end{aligned} \tag{2.12}$$

$$D_s F_\epsilon|_{\epsilon=0} = \eta, \tag{2.13}$$

$$\frac{\partial}{\partial \epsilon} F_\epsilon = -\eta^2 \int_0^T (1 + \epsilon S_t^2) S_t^2 dt + \eta \int_0^T S_t^2 dB_t, \tag{2.14}$$

$$\gamma(F_\epsilon)|_{\epsilon=0} = \eta^2 T.$$

This example will be applied for the generalized Hobson-Rogers model (GHR).

Example 2. *A more sophisticated example is given by the following structure, which resembles a slightly modified version of the original Hobson-Rogers model (HR):*

$$F_\epsilon := z - \frac{1}{2} \int_0^T \sigma(S_t)^2 dt + \int_0^T \sigma(S_t) dB_t, \tag{2.15}$$

$$dS_t = (-\frac{1}{2}\sigma(S_t)^2 - \lambda S_t)dt + \sigma(S_t)dB_t \tag{2.16}$$

with $\sigma(s) = \eta\sqrt{1+\epsilon s^2}\exp(-\frac{\epsilon^2 s^2}{M})$ *for some large constant* M. *Hence* σ *is* C^∞-*bounded and bounded and we obtain* $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ *(see* [18] *or* [21]*). The relevant derivatives at* $\epsilon = 0$ *read as follows.*

$$\begin{aligned}
&D_s F_\epsilon|_{\epsilon=0} = \eta,\\
&\frac{\partial}{\partial\epsilon} F_\epsilon|_{\epsilon=0} = -\int_0^T \frac{\eta^2 S_t^2|_{\epsilon=0}}{2} dt + \int_0^T \frac{\eta S_t^2|_{\epsilon=0}}{2} dB_t,\\
&\gamma(F_\epsilon)|_{\epsilon=0} = \eta^2 T,
\end{aligned}$$

where $(S_t)_{t\geq 0}$ *at* $\epsilon = 0$ *is particularly simple, namely a mean-reverting Gaussian process,*

$$dS_t = (-\frac{1}{2}\eta^2 - \lambda S_t)dt + \eta dB_t.$$

Theorem 1. *Given a family* $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ *with regular density. Then there exist random variables* $\pi^m \in \mathcal{D}^\infty$ *such that for all* $\phi \in C_0^\infty(\mathbb{R})$

$$\frac{\partial^m}{\partial\epsilon^m} E(\phi(F_\epsilon))|_{\epsilon=0} = E(\phi(F_0)\pi^m) \tag{2.17}$$

holds true for $n \geq 0$.

Definition 2. *The random variable* $\pi^n \in \mathcal{D}^\infty$ *is called* n*th Malliavin weight for differentiation with respect to the parameter* ϵ.

Proof. We fix $m \geq 0$. For the general construction of the weights we apply the Faa-di-Bruno formula (see for instance [6]), which calculates the coefficients of the series

$$\frac{\partial^m}{\partial\epsilon^m}\phi(F_\epsilon) = \sum_{n=0}^m \phi^{(n)}(F_\epsilon) G_\epsilon^n,$$

where in G_ϵ^n polynomials of derivatives of F_ϵ appear. We apply multi-index notation here,

$$G_\epsilon^{n,m} = \sum_{\substack{\lambda_\alpha \in \mathbb{N}^{\mathbb{N}\setminus\{0\}} \\ \sum_{\alpha\in\mathbb{N}\setminus\{0\}} \lambda_\alpha = n \\ \sum_{\alpha\in\mathbb{N}\setminus\{0\}} \alpha\lambda_\alpha = m}} \frac{m!}{\lambda!} \prod_{\alpha\in\mathbb{N}\setminus\{0\}} (\frac{1}{\alpha}\frac{\partial^\alpha}{\partial\epsilon^\alpha} F_\epsilon)^{\lambda_\alpha}$$

The following simple consideration, for $G_0^n \in \mathcal{D}^\infty$ then yields the result,

$$\begin{aligned}
E(\phi^{(n)}(F_0)G_0^n) &= E(\phi^{(n)}(F_0)\int_0^T \frac{D_s F_0 D_s F_0}{\gamma(F_0)} ds G_0^n)\\
&= E(\int_0^T D_s\phi^{(n-1)}(F_0)\frac{D_s F_0}{\gamma(F_0)} G_0^n ds)\\
&= E(\phi^{(n-1)}(F_0)\delta(s \mapsto \frac{D_s F_0}{\gamma(F_0)} G_0^n))
\end{aligned}$$

for $m \geq n \geq 1$. By induction we can prove the result. □

We define – in view of the previous proof – the following iterative procedure: let $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ be a family with regular density, then

$$\operatorname{div}^F(G) := \delta(\mapsto \frac{D_s F_0}{\gamma(F_0)} G) \tag{2.18}$$

for $G \in \mathcal{D}^\infty$, which is an element of $\mathcal{D}^\infty$ by property 4. of the primer.

Lemma 1. *Let $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ be family with regular density, such that F_0 is Gaussian, i.e. $D_s F_0$ is deterministic and the covariance matrix $\gamma(F_0)$ is constant. Let furthermore $P_u^{(j)} \in R[\mathbb{R}^n]$, for $j = 1, 2$ be polynomials with continuously time-dependent coefficients on $[0,T]$ in n variables, and let $(X_u)_{0\leq u\leq T} = (g_1(u) + \int_0^T f_1(u,t)dB_t, \dots, g_n(u) + \int_0^T f_n(u,t)dB_t)_{0\leq s\leq T}$ be continuous Gaussian process in $\mathbb{R}^n$, where g_i and f_i are continous functions on the respective domains. Then the following assertions hold.*

(1) *The random variable $G := \int_0^T P_u^{(1)}(X_u)du \int_0^T P_u^{(2)}(X_u)du \in \mathcal{D}^\infty$, in fact arbitrary polynomials in such random variables are smooth.*
(2) *The F-divergence of G is of the same form*

$$\begin{aligned}\operatorname{div}^F(G) = &\int_0^T P_u^{(1)}(X_u)du \int_0^T P_u^{(2)}(X) \int_0^T \frac{D_t F_0}{\gamma(F_0)} dB_t du - \\ &- \sum_{i=1}^n \int_0^T \int_0^T \frac{\partial}{\partial x^i} P_u^{(1)}(X) f_i(u,t) \frac{D_t F_0}{\gamma(F_0)} dt du \int_0^T P_u^{(2)}(X_u)du + \\ &+ \sum_{i=1}^n \int_0^T \int_0^T \frac{\partial}{\partial x^i} P_u^{(2)}(X) f_i(u,t) \frac{D_t F_0}{\gamma(F_0)} dt du \int_0^T P_u^{(1)}(X_u)du.\end{aligned}$$

Proof. The integral exists almost surely and satisfies by Hölder estimates all necessary estimates. The continuous process $P_u^{(1)}(X_u)$ lies in the sum of the first n chaos subspaces point by point in u, and the integral, too, by closedness and the polynomial property of $\mathcal{D}^\infty$, hence the assertion of 1. The second formula follows by applying the F-divergence point by point in u to the element G by property 4 of the primer. □

Remark 1. *A random variable $\int_0^T P_u^{(1)}(X_u)du$ is called an* integrated Gaussian polynomial*.*

Theorem 2. *Let $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ be a family with regular density, such that F_0 is Gaussian, i.e. $D_s F_0$ is deterministic and the covariance matrix $\gamma(F_0)$ is constant. Furthermore we assume that the derivatives $\frac{\partial^n}{\partial \epsilon^n} F_\epsilon|_{\epsilon=0}$ are integrated Gaussian polynomials. Then $\pi^n \in \mathcal{D}^\infty$ is a polynomial of integrated Gaussian polynomials.*

Proof. By the Faa-di-Bruno formula G_0^n is a polynomial of integrated Gaussian polynomials, hence by inductive use of Lemma 1 we obtain that π^n is a polynomial of integrated Gaussian polynomials. □

Remark 2. *If π^n is a polynomial of integrated Gaussian polynomials, then the expected value $E(\phi(F_0)\pi^n)$ can be calculated in two steps: first an ordinary Gaussian integral applied to a polynomial on some $\mathbb{R}^n$, second the inegration of this result with respect to Lebesgue measure on $[0,T]^m$. Both procedures are numerically cheap and yield quick and good results even for complicated stochastic differential equations.*

The applications which we have in mind are certainly solutions of standard stochastic differential equations of the type

$$dZ_t^{x,\epsilon} = V(\epsilon, t, Z_t^{x,\epsilon})dt + V^1(\epsilon, t, Z_t^{x,\epsilon})dB_t, \tag{2.19}$$

where the initial value is given by a real vector $x \in \mathbb{R}^N$ and $(B_t)_{t\geq 0}$ denotes a 1-dimensional Brownian motion. If the vector fields V, V^1 are regular enough, for instance real analytic and C^∞-bounded, then we can take each coordinate of the solution process at a certain time $T > 0$ – viewed as a family of random variables with respect to $\epsilon \geq 0$ – is an element of $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$.

Corollary 1. *Let $(X_T^{x,\epsilon})_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ be a component of the solution process of equation 2.19, which is a family with regular density. Take a bounded real analytic function with bounded derivatives $\phi : \mathbb{R} \to \mathbb{R}$, then*

$$E(\phi(X_T^{x,\epsilon})) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} E(\phi(X_T^{x,0})\pi^n)$$

for small $\epsilon > 0$ (the size of the neighborhood might depend on t and x).

Proof. By the Cauchy-Kowalewsky Theorem we know the the associated parabolic initial value problem has a real analytic solution, which coincides a fortiori with $E(\phi(X_t^{x,\epsilon}))$. Hence the Taylor series converges locally and the above representation holds. □

If we want to prove the convergence of the series for more general payoff functions ϕ, we can apply the following sufficient conditions.

Theorem 3. *Let $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ be a family with regular density. Assume furthermore that the (universally calculated) weights π^n satisfy*

$$\sum_{n\geq 0} \frac{\epsilon_0^n}{n!} E((\pi^n)^2)^{\frac{1}{2}} < \infty$$

for some $\epsilon_0 > 0$, then for all bounded measurable $\phi : \mathbb{R} \to \mathbb{R}$ we obtain

$$E(\phi(F_\epsilon)) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} E(\phi(F_0)\pi^n)$$

for $\epsilon < \epsilon_0$.

Proof. Take a sequence of bounded real analytic ϕ_k with bounded derivatives such that $E((\phi_k - \phi)^2(F_0)) \to 0$ as $k \to \infty$, then by the Cauchy-Schwarz inequality

$$\begin{aligned} |E(\phi_l(F_\epsilon) - \phi_k(F_\epsilon))| &\leq \sum_{n\geq 0} \frac{\epsilon^n}{n!} |E((\phi_k(F_0) - \phi_l(F_0))\pi^n)| \\ &\leq E((\phi_k - \phi_l)^2(F_0))^{\frac{1}{2}} \sum_{n\geq 0} \frac{\epsilon_0^n}{n!} E((\pi^n)^2)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $k, l \to \infty$ on $[0, \epsilon_0]$ uniformly in ϵ. The same holds for all derivatives and hence we can conclude that $\epsilon \mapsto E(\phi(F_\epsilon))$ is real analytic and the series above yields the correct power series expansion. □

A particular feature of the considerations above is, that we do not need the integrability assumptions on the Malliavin covariance matrix off $\epsilon = 0$. If we are able to calculate the π^n (which involve terms at $\epsilon = 0$) and prove a regularity assumption like $\sum_{n\geq 0} \frac{\epsilon_0^n}{n!} E((\pi^n)^2)^{\frac{1}{2}} < \infty$, we are able to show real analyticity of the above type for $E(\phi(F_\epsilon))$.

Similar reasonings can be applied for the calculation of the Greeks: here we consider precisely the same setting, only in a two-dimensional framework, since we consider the process $(X_t^{x,\epsilon}, \frac{d}{dx} X_t^{x,\epsilon})_{t\geq 0}$ and the respective initial values. By the Cauchy-Kowalewsky Theorem we are able to conclude that for real analytic ϕ with bounded derivatives the expansion

$$\frac{d}{dx} E(\phi(X_t^{x,\epsilon})) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} E(\phi(X_t^{x,0})\rho^n)$$

converges, where the ρ^n are again given by Skorohod integrals. Finally also the considerations with respect to bounded measurable functions apply.

Remark 3. *If we know from semigroup considerations (analyticity with respect to parameters) that for all bounded measurable ϕ the series*

$$E(\phi(X_t^{x,\epsilon})) = \sum_{n\geq 0} \frac{\epsilon^n}{n!}\frac{\partial^n}{\partial\epsilon^n} E(\phi(X_t^{x,\epsilon}))|_{\epsilon=0}$$

converges for small ϵ, then we can conclude that

$$\frac{\partial^n}{\partial\epsilon^n} E(\phi(X_t^{x,\epsilon}))|_{\epsilon=0} = E(\phi(X_t^{x,0})\pi^n),$$

holds true for $n \geq 0$, if a uniform convergence – of the respective derivatives with real analytic ϕ to derivatives for bounded measurable ones – applies

Example 3. *We take (2.11) with $a = \lambda = 0$ and $b = 1$, $S_0 = 0$, and calculate the outcome for the first and second derivative with respect to ϵ.*

$$\begin{aligned}
E(\phi(F_\epsilon)) &= E(\phi(z - \frac{\eta^2}{2}T + \eta B_T)) \\
\frac{\partial}{\partial\epsilon}|_{\epsilon=0} E(\phi(F_\epsilon)) &= E(\phi(z - \frac{\eta^2}{2}T + \eta B_T)\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon \frac{\eta}{\eta^2 T})) \\
&= \frac{1}{\eta T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) B_T(-\eta^2 \int_0^T B_t^2 dt + \eta \int_0^T B_t^2 dB_t)) + \\
&+ \frac{1}{\eta T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) \int_0^T (-2\eta^2 \int_s^T B_t dt + \eta B_s^2 + 2\eta \int_s^T B_t dB_t) ds) \\
&= \frac{1}{T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) B_T(\frac{B_T^3}{3} - \int_0^T B_t dt - \eta \int_0^T B_t^2 dt)) + \\
&+ \frac{1}{T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T)(B_T^2 T - \frac{T^2}{2} - 2\eta \int_0^T \int_s^T B_t dt ds)),
\end{aligned}$$

which has a simple integrated polynomial structure in Gaussian random variables.

For the second derivative we proceed as follows: We observe that two ingredients for Skorohod integral can be well-calculated, namely

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}|_{\epsilon=0} \delta(\frac{\partial}{\partial\epsilon} F_\epsilon \frac{D_s F_\epsilon}{\gamma(F_\epsilon)}) &= \delta(\frac{\partial^2}{\partial\epsilon^2}|_{\epsilon=0} F_\epsilon \frac{1}{\eta T}) + \\
+ \delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon \frac{D_s \frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon}{\eta^2 T}) &- \delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon \frac{\eta}{\eta^4 T^2} \frac{\partial}{\partial\epsilon}|_{\epsilon=0} \gamma(F_\epsilon)),
\end{aligned}$$

and

$$\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon \frac{\eta}{\eta^2 T} \delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0} F_\epsilon \frac{\eta}{\eta^2 T}))$$

as above. Again we shall obtain a simple polynomial structure.

3. Generalised Hobson-Rogers models

In this section we shall provide the basic notions for the simplest specification of the Hobson-Rogers model, its generalisation and the analytically tractable approximation procedure as proposed by Theorem 1, which serves for estimating the parameters easily. In particular both models satisfy the hypothese of Theorem 2, hence the weights π^n can be easily calculated.

3.1. **The Hobson-Rogers Model.** Hobson and Rogers proposed in [12] a complete stochastic volatility model with offset functions. We shall only consider models where we allow the first offset function and we fix a certain dynamics for this model, which was also proposed in [12]. Fix a time horizon $T > 0$. Given a price process $(P_t)_{0\leq t\leq T}$, which is a positive, square integrable Ito process on a stochastic basis $(\Omega, \mathcal{F}_T, P)$ with one-dimensional Brownian motion $(B_t)_{0\leq t\leq T}$, we introduce $Z_t := \ln(e^{-rt}P_t)$, where $r \geq 0$ denotes the interest rate. Then we assume for a positive parameter $\lambda > 0$ the 2-dimensional Markov process (Z_t, S_t)

$$dZ_t = \mu(S_t)dt + \sigma(S_t)dB_t \tag{3.1}$$

$$dS_t = dZ_t - \lambda S_t dt = (\mu(S_t) - \lambda S_t)dt + \sigma(S_t)dB_t, \tag{3.2}$$

$$Z_0 = z, S_0 = s, \tag{3.3}$$

for volatility and drift vector fields μ, σ which satisfy the usual Lipschitz assumptions and $\sigma(s) > 0$ for $s \in \mathbb{R}$. The process $(S_t)_{0\leq t\leq T}$ is the first offset function of $(Z_t)_{0\leq t\leq T}$ with parameter $\lambda > 0$. Defining

$$\theta(s) = \frac{1}{2}\sigma(s) + \frac{\mu(s)}{\sigma(s)},$$

where we additionally assume that the measure Q_t on $\mathcal{F}_t$ given through

$$\frac{dQ_t}{dP} = \exp(-\int_0^t \theta(S_u)dB_u - \frac{1}{2}\int_0^t \theta(S_u)^2 du)$$

is well-defined for $0 \leq t \leq T$ and that $Q := Q_T$ is a probability measure equivalent to P on $\mathcal{F}_T$ (see [9] for a nice elaboration of the relevant conditions). Then the process $\widetilde{B_t} := B_t + \int_0^t \theta(S_u)du$ is a Q-Brownian motion and the stochastic differential equation reads as follows with respect to $(\widetilde{B_t})_{0\leq t\leq T}$

$$\begin{aligned} dZ_t &= -\frac{1}{2}\sigma(S_t)^2 dt + \sigma(S_t)d\widetilde{B_t}, \\ dS_t &= -(\frac{1}{2}\sigma(S_t)^2 + \lambda S_t)dt + \sigma(S_t)d\widetilde{B_t}, \\ Z_0 &= z, S_0 = s \end{aligned}$$

for $0 \leq t \leq T$.

The discounted price process $(e^{-rt}P_t)_{0\leq t\leq T}$ is a Q-martingale and we can apply the classical no-arbitrage pricing arguments. In particular the market is complete since this is the only martingale measure equivalent to P. Under Q the price process satisfies

$$dP_t = rP_t dt + \sigma(S_t)P_t d\widetilde{B_t}$$

Therefore the price of a European claim, which is given by a measurable function with at most linear growth $q : \mathbb{R} \to \mathbb{R}$, is defined by

$$V(P_t, S_t, T-t) = e^{r(T-t)}E(q(P_T)|\mathcal{F}_t)$$

for $0 \leq t \leq T$ via the Markov property. If the Lie algebra spanned by the two vector fields

$$\begin{aligned} (z,s) &\mapsto \begin{pmatrix} \sigma(s) \\ \sigma(s) \end{pmatrix} \\ (z,s) &\mapsto \begin{pmatrix} -\frac{1}{2}\sigma(s)^2 - \frac{1}{2}\sigma(s)'\sigma(s) \\ -\frac{1}{2}\sigma(s)^2 - \lambda s - \frac{1}{2}\sigma(s)'\sigma(s) \end{pmatrix} \end{aligned}$$

or equivalently

$$(z,s) \mapsto \begin{pmatrix} \sigma(s) \\ \sigma(s) \end{pmatrix} \text{ and } (z,s) \mapsto \begin{pmatrix} 0 \\ \lambda s \end{pmatrix}$$

spans the tangent space $\mathbb{R}^2$ pointwise on $\mathbb{R}_{>0} \times \mathbb{R}$ (which is the case for non-vanishing σ and $\lambda \neq 0$), then by Hörmander's "Sum of the Squares" we know that V is a smooth function on $\mathbb{R}_{>0} \times \mathbb{R} \times]0, T[$ and satisfies the boundary condition $V(p, s, 0) = q(p)$ for all $(p, s) \in \mathbb{R}_{>0} \times \mathbb{R}$.

One particular choice for σ proposed in [12] is a smooth vector field $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(s) := \eta\sqrt{1 + \epsilon s^2} \tag{3.4}$$

on some ball with large radius $R > 0$ and constant outside for fixed $\epsilon \geq 0$. Here η is referred to as minimal level of implied volatility and $\epsilon \geq 0$ denotes a parameter calibrating the influence of the first offset process $(S_t)_{t\geq 0}$ on the stochastic evolution of the price process. The results cited in Section 2 stem from this model. In the sequel we shall refer to this model as Hobson-Rogers model (3.4). Furthermore the option price depends smoothly on the parameters η, ϵ and λ on the respective intervals of definition (see for instance [14]). By standard methods we can find a version of the solution $(Z_t, S_t)_{0\leq t\leq T}$ of the stochastic differential equation, which depends in a smooth way on the initial values and the parameters. Hence by dominated convergence we obtain families $(Z_T^{(\epsilon)})_{\epsilon\geq 0}$ of random variables in $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$.

3.2. **The Generalised Hobson-Rogers Model (GHR).** We shall only calculate with respect to the equivalent martingale measure Q in view of option pricing. In view of the unsatisfying curvature of the smiles for the model (3.4) we propose the following generalization:

$$dZ_t = -\frac{1}{2}\sigma_1(S_t)^2 dt + \sigma_1(S_t) d\widetilde{B_t} \tag{3.5}$$

$$dS_t = \mu(S_t) dt + \sigma_2(S_t) d\widetilde{B_t} \tag{3.6}$$

$$Z_0 = z, S_0 = s \tag{3.7}$$

with the following specification,

$$\sigma_1(s) = \eta(1 + \epsilon\beta s^2) \tag{3.8}$$

$$\sigma_2(s) = \chi\eta \tag{3.9}$$

$$\mu(s) = -\frac{\eta^2}{2} - \lambda s \tag{3.10}$$

for fixed $\epsilon \geq 0$. In contrast to (3.4) we are additionally given two positive parameters $\chi \geq 1$ and $\beta \in [0, \frac{1}{2}]$ (even though only the product $\epsilon\beta$ enters into the formulas). By Hörmander's Theorem we can analogously conclude that the option price depends smoothly on the initial values. Furthermore the option price depends smoothly on the parameters η, ϵ and λ and the respective intervals of definition. By standard methods we can find a version of the solution (Z_t, S_t) of the stochastic differential equation, which depends smoothly on the initial values and the parameters, which we shall assume in the sequel.

3.3. **Analytically tractable approximations for GHR.** We can calculate the solution of GHR directly

$$dZ_t^{(\epsilon)} = -\frac{1}{2}\eta^2(1 + \epsilon\beta S_t^2)^2 dt + \eta(1 + \epsilon\beta S_t^2) d\widetilde{B_t} \tag{3.11}$$

$$dS_t = (-\frac{\eta^2}{2} - \lambda S_t) dt + \chi\eta d\widetilde{B_t}. \tag{3.12}$$

For $0 \leq t \leq T$ the curve $\epsilon \mapsto Z_t^{(\epsilon)}$ lies in $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$, which follows by calculating the Malliavin derivatives and the derivatives with respect to ϵ. Furthermore, the Malliavin covariance matrix is invertible for $t > 0$ with inverse bounded with respect to all L^p-norms at $\epsilon = 0$.

The solution on $[0,T]$ is given by

$$S_t = e^{-\lambda t}s - \int_0^t e^{-\lambda(t-u)}\frac{\eta^2}{2}du + \int_0^t \chi\eta e^{-\lambda(t-u)}d\widetilde{B_u},$$

which is a Gaussian process. We can express the Brownian motion $(\widetilde{B_t})_{t\geq 0}$ by this process,

$$d\widetilde{B_t} = \frac{1}{\chi\eta}(\frac{\eta^2}{2} + \lambda S_t)dt + \frac{1}{\chi\eta}dS_t,$$

which leads to

$$\begin{aligned} dZ_t^{(\epsilon)} &= -\frac{1}{2}\eta^2(1+\epsilon\beta S_t^2)^2dt + \frac{1}{\chi}(1+\epsilon\beta S_t^2)(\frac{\eta^2}{2}+\lambda S_t)dt + \frac{1}{\chi}(1+\epsilon\beta S_t^2)dS_t \\ &= (-\frac{1}{2}\eta^2 + \frac{\eta^2}{2\chi})dt + \frac{\lambda}{\chi}S_tdt + \frac{1}{\chi}dS_t + \\ &+ \epsilon(\frac{\eta^2\beta(1-2\chi)}{2\chi}S_t^2dt + \frac{\lambda\beta}{\chi}S_t^3dt + \frac{\beta}{3\chi}dS_t^3 - \beta\eta^2\chi S_tdt) + \\ &- \frac{\epsilon^2}{2}\beta^2\eta^2S_t^4dt. \end{aligned}$$

This can be applied to the calculation of the first and second variation at $\epsilon = 0$ of the process $(Z_t^{(\epsilon)})_{0\leq t\leq T}$:

$$\begin{aligned} Z_t^{(0)} &= z + (-\frac{1}{2}\eta^2 + \frac{\eta^2}{2\chi})t + \int_0^t \frac{\lambda}{\chi}S_udu + \frac{1}{\chi}S_t \\ \frac{\partial}{\partial\epsilon}Z_t^{(\epsilon)}|_{\epsilon=0} &= \frac{\eta^2\beta(1-2\chi)}{2\chi}\int_0^t S_u^2du + \frac{\lambda\beta}{\chi}\int_0^t S_u^3du + \frac{\beta}{3\chi}S_t^3 - \\ &- \frac{\beta}{3\chi}s^3 - \beta\eta^2\chi\int_0^t S_udu, \\ \frac{\partial^2}{\partial\epsilon^2}Z_t^{(\epsilon)}|_{\epsilon=0} &= -\int_0^t \beta^2\eta^2S_u^4du. \end{aligned}$$

We have proved the following Theorem:

Theorem 4. *The weights π^n for the GHR model are polynomials of integrated Gaussian polynomials.*

Proof. The process $Z_t^{(0)}$ is a Gaussian process and the covariance matrix is invertible. Furthermore derivatives $\frac{\partial^n}{\partial\epsilon^n}Z_t^{(\epsilon)}|_{\epsilon=0}$ are seen to be integrated Gaussian polynomials. Hence we apply Theorem 2. □

Remark 4. *We could have chosen*

$$\sigma(s) = \eta(1+\epsilon s^2)^\beta \tag{3.13}$$

to obtain the same first two variations at $\epsilon = 0$, which expresses the fact that for $\beta = \frac{1}{2}$ and $\chi = 1$ we are near the Hobson-Rogers model.

The pricing of a European call option is then given by

$$\begin{aligned} V(z,s,K,T,r,\epsilon,\lambda,\chi,\eta,\beta) &= E(e^{-rT}(P_T-K)_+) \\ &= KE(e^{-rT}(\frac{P_T}{K}-1)_+) \\ &= KE(e^{-rT}1_{\{\frac{P_T}{K}-1\geq 0\}}(\frac{P_T}{K}-1)). \end{aligned}$$

This is the precise price, for which we try to find an analytically tractable approximation.

We shall apply the Taylor expansion from Theorem 2 up to first order. We know by this theorem that the weights will be polynomials of inegrated Gaussian polynomials. We can calculate the derivative directly, avoiding the calculation of the Skorohod integral, since

$$\frac{D_s Z_T^{(0)}}{\gamma(Z_T^{(0)})} = \frac{1}{\eta T}$$

for $0 \leq s \leq T$, hence

$$\begin{aligned} E(\phi(Z_T^{(0)})\delta(\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_T^{(\epsilon)} \frac{D_s Z_T^{(\epsilon)}}{\gamma(Z_T^{(0)})})) &= E(\int_0^T \phi'(Z_T^{(0)})\eta 1_{[0,T]}(s)\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_T^{(\epsilon)} \frac{1}{\eta T} ds) \\ &= E(\phi'(Z_T^{(0)})\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_T^{(\epsilon)}). \end{aligned}$$

So we conclude by

$$V(z,s,K,T,r,0,\lambda,\chi,\eta,\beta) = KE(e^{-rT} 1_{\{\frac{\exp(rT+Z_T^{(0)})}{K}-1\geq 0\}}(\frac{\exp(rT+Z_T^{(0)})}{K}-1)_+) \tag{3.14}$$

$$\frac{\partial}{\partial \epsilon}|_{\epsilon=0} V = KE(e^{-rT} 1_{\{\frac{\exp(rT+Z_T^{(0)})}{K}-1\geq 0\}} \frac{\exp(rT+Z_T^{(0)})}{K}\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_T^{(\epsilon)}) \tag{3.15}$$

$$\begin{aligned} &= \frac{\beta}{3\chi} E(1_{\{e^{rT+Z_T^{(0)}}\geq \ln K\}} e^{Z_T^{(0)}}(S_T^3 - s)) + \\ &+ \int_0^T E(1_{\{e^{rT+Z_T^{(0)}}\geq 1\}} e^{Z_T^{(0)}}(\frac{\lambda\beta}{\chi}S_u^3 + \frac{\eta^2\beta(1-2\chi)}{2\chi}S_u^2 - \beta\chi\eta^2 S_u))du. \end{aligned} \tag{3.16}$$

Surprisingly the Taylor formula up to first order actually is a good approximation for the solution for certain parameter intervals, which can be proved by a (lengthy) estimation argument or which can be seen directly by numerical justification.

The polynomial structure of the weights π^1 gets immediately visible, as proved in Theorem 2. The formula decomposes in the well-known Black-Scholes formula as the first term and a second term which is given by integration with respect to the following two-dimensional Gaussian variable: we denote by $(Z_t^{(0)}, S_t)_{0\leq t\leq T}$ the solutions at $\epsilon = 0$ with initial values (z, s).

$$\begin{aligned} Z_T^{(0)} &= z - \frac{\eta^2}{2}T + \eta\widetilde{B_T}, \\ S_s &= e^{-\lambda s}S_0 - \int_0^s e^{-\lambda(s-u)}\frac{\eta^2}{2}du + \int_0^s e^{-\lambda(s-u)}\chi\eta d\widetilde{B_u}, \end{aligned}$$

with expectations and covariances,

$$\begin{aligned} E(Z_T^{(0)}) &= z - \frac{\eta^2}{2}T, \\ E(S_v) &= e^{-\lambda v}(v - \frac{\eta^2}{2}\frac{e^{\lambda v}-1}{\lambda}), \\ E((Z_T^{(0)} - E(Z_T^{(0)}))^2) &= \eta^2, \\ E((S_v - E(S_v))^2) &= \eta^2\chi^2 e^{-2\lambda v}\frac{e^{2\lambda v}-1}{2\lambda}, \\ \operatorname{cov}(Z_T^{(0)}, S_v) &= \int_0^v \eta^2\chi e^{-\lambda(v-u)}du, \\ &= \eta^2\chi e^{-\lambda v}\frac{e^{\lambda v}-1}{\lambda} \end{aligned}$$

for $0 \leq v \leq T$. These formulas have been used in order to estimate the parameters for the graphs in Section 1.

3.4. **Analytically tractable approximations for the HR model.** The variational equation for derivatives with respect to ϵ at $\epsilon = 0$ are given by

$$d\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_t^{(\epsilon)} = -\frac{\eta^2}{2} S_t^2 dt + \frac{\eta}{2} S_t^2 d\widetilde{B_t} \tag{3.17}$$

$$dS_t = (-\frac{\eta^2}{2} - \lambda S_t)dt + \eta d\widetilde{B_t}. \tag{3.18}$$

This is analytically tractable: the solution of 3.18 is given on $[0, T]$ through

$$S_t = e^{-\lambda t} s - \int_0^t e^{-\lambda(t-u)} \frac{\eta^2}{2} du + \int_0^t \eta e^{-\lambda(t-u)} d\widetilde{B_u},$$

which is a Gaussian process. We can express the Brownian motion $(\widetilde{B_t})_{t \geq 0}$ by this process,

$$d\widetilde{B_t} = \frac{1}{\eta}(\frac{\eta^2}{2} + \lambda S_t)dt + \frac{1}{\eta} dS_t,$$

which leads to

$$\begin{aligned} d\frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_t &= -\frac{1}{2}\eta^2 S_t^2 dt + \frac{1}{2} S_t^2 (\frac{\eta^2}{2} + \lambda S_t)dt + \frac{1}{2} S_t^2 dS_t \\ &= -\frac{\eta^2}{4} S_t^2 dt + \frac{\lambda}{2} S_t^3 dt + \frac{1}{6} dS_t^3 - \frac{1}{2}\eta^2 S_t dt. \end{aligned}$$

This can be applied to the calculation of the first variation:

$$\begin{aligned} \frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_t = -\frac{\eta^2}{4} \int_0^t S_u^2 du + \frac{\lambda}{2} \int_0^t S_u^3 du + \frac{1}{6} S_t^3 - \\ -\frac{1}{6} s^3 - \frac{1}{2}\eta^2 \int_0^t S_u du. \end{aligned}$$

The pricing of a European call option is then given by

$$\begin{aligned} V(z, s, K, T, r, \epsilon, \lambda, \chi, \eta, \beta) &= E(e^{-rT}(P_T - K)_+) \\ &= KE(e^{-rT}(\frac{P_T}{K} - 1)_+) \end{aligned}$$

and the first derivative with respect to ϵ

$$\begin{aligned} \frac{\partial}{\partial \epsilon}|_{\epsilon=0} V(z, s, K, T, r, \epsilon, \lambda, \chi, \eta, \beta) &= KE(e^{-rT} 1_{\{\frac{e^{rT+Z_T^{(0)}}}{K} - 1 \geq 0\}} \frac{e^{rT+Z_T}}{K} \frac{\partial}{\partial \epsilon}|_{\epsilon=0} Z_T) \\ &= \frac{1}{6} E(1_{\{e^{rT+Z_T^{(0)}} \geq \ln K\}} e^{Z_T^{(0)}} (S_T^3 - s^3)) + \\ &+ \int_0^T E(1_{\{e^{rT+Z_T} \geq 1\}} e^{Z_T^{(0)}} (\frac{\lambda}{2} S_u^3 - \frac{\eta^2}{4} S_u^2 - \frac{1}{2}\eta^2 S_u))du, \end{aligned}$$

which therefore yields the same first derivative at $\epsilon = 0$ as in equation 3.15.

Theorem 5. *The weights π^1 and π^2 for the HR model are polynomials of integrated Gaussian polynomials.*

Proof. The process $Z_t^{(0)}$ is a Gaussian process and the covariance matrix is invertible. Furthermore derivatives $\frac{\partial^n}{\partial \epsilon^n} Z_t^{(\epsilon)}|_{\epsilon=0}$ for $n = 0, 1, 2$ are seen to be integrated Gaussian polynomials. Hence we apply a version of Theorem 2.notice that $G_0 = 0$. □

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