Habilitationsschrift

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Contents

1. Introduction

The purpose of this Habilitationsschrift, which consists of a collection of recent research papers published in refereed scientific journals, is the description of my research and the description of the interplay between two distinguished research areas, namely Infinite dimensional Differential Geometry and Interest Rate Theory. I have chosen a deductive rather than an inductive presentation, that means I shall present the latest articles first and finish by those related to my thesis. The latest articles are emphasized in the presentation since the concepts under consideration appear to be more recent.

The structure of the Habilitationsschrift is the following: the subsections of the overview section are related to the successive chapters, where the articles are presented. The bibliographies are independently attached for each paper and also for the introductory section. If the cited paper can be found in the Habilitationsschrift, it will be indicated directly in the citation by the abbreviation *HS*. I follow the good tradition that each mathematical text should at least contain one original result together with its proof, so the overview section contains an unpublished new version of the Frobenius Theorem.

2. Overview

In this section the interplay between Infinite dimensional Geometry and Stochastic Analysis, in particular Interest Rate Theory, is described. On the one hand geometry is a universal science, whose methods can be applied in several mathematical areas. It is not surprising that it might be interesting to describe the geometry of stochastic flows on infinite dimensional Hilbert spaces. On the other hand recent progress in infinite dimensional geometry, namely the discovery of convenient calculus and the convenient formulation of differential geometry (see [KrieglMichor1997]), opened new perspectives on several geometric problems. What has been treated so far on Sobolev hierarchies with a careful bookkeeping of analytic properties, can be treated now by simpler and more elegant methods on spaces, which are certainly more adapted to the problem in mind. This is the real progress, which makes infinite dimensional geometry a natural tool. I have learned these reasonings during the work on my thesis: the credo that analysis and geometry beyond Banach spaces is powerful represents the link between my two research areas.

2.1. Finite dimensional Realizations. Recall first the finite dimensional situation: given $U \subset \mathbb{R}^n$, $n \geq 1$, an open, connected set and a finite dimensional *distribution* on U, i.e. a collection of vector spaces $D_u \subset \mathbb{R}^n$ for each $u \in U$. We assume that $\mathcal{D} = (D(u))_{u \in U}$ is *smooth*, that is for each point $u_0 \in U$ there are smooth vector fields $X_1, \dots, X_m : V \subset U \to \mathbb{R}^n$ such that

$$D(u) = \langle X_1(u), \dots, X_m(u) \rangle \tag{2.1}$$

for all $u \in V$, here V is an open neighborhood of $u_0 \in U$. If $X_1, \dots, X_m$ are pointwise linearly independent, then it this is called a *local frame*. A submanifold $M \subset U$ is called *integral* if $T_m M = D(m)$ for all $m \in M \subset U$.

Example 1. *Take vector fields $X_1, \dots, X_m : U \to \mathbb{R}^n$, then the span 2.1 is a distribution and integral submanifolds are submanifolds pointwise tangent to X_i for $i = 1, \dots, m$.*

Given a smooth distribution $\mathcal{D}$ of constant rank, i.e. $\dim D(u)$ is locally constant, the *Frobenius Theorem* asserts that any point in U is contained in an integral submanifold if and only if the distribution is involutive. A distribution is called *involutive* if for all smooth vector fields $X, Y : V \subset U \to \mathbb{R}^n$, which take values in $\mathcal{D}$, the Lie bracket also takes values in $\mathcal{D}$

$$[X,Y](u) := DX(u)\cdot Y(u) - DY(u)\cdot X(u) \in D(u)$$

for all $u \in V$. This nice Theorem can be easily generalized to Banach spaces.

Morally the Frobenius Theorem relates the algebraic fact that a distribution is involutive to the (analytic) existence of integral submanifolds. Up to Banach spaces the only analytic requirement is smoothness or comparable analytic properties such as C^n, which is even necessary to calculate the Lie brackets.

Nevertheless modern, interesting analytic problems related to the existence of integral submanifolds appear often under circumstances, where this basic analytic property fails on Banach spaces. I think of distributions generated by infinitesimal generators of strongly continuous semigroups on Banach spaces or the question of existence of strongly continuous, unitary representations on Hilbert spaces. In the latter case we are given the following situation on a complex Hilbert space H: let $\frac{1}{i}A_1, \ldots, \frac{1}{i}A_m$ be self-adjoint operators and assume that there is a dense subspace $D \subset H$ such that $A_i|_D : D \to D$ are well-defined linear operators for $i = 1, \ldots, m$ and

$$[A_i, A_j] = \sum_{k=1}^{n} c_{ij}^k A_k \tag{2.2}$$

for some real (structure) constants c_{ij}^k and $i, j = 1, \ldots, m$. For the existence of integral submanifolds tangent to the distribution $D_u = \langle A_1 u, \ldots, A_n u\rangle$ for $u \in D$ one needs additional analytic requirements such as Nelson's famous condition on the Casimir element $\sum_{i=1}^m A_i^2$ to be essentially self-adjoint on D. This typical situation explains why the classical Frobenius Theorem formulated on Banach spaces is not sufficient for many infinite dimensional problems.

In [Teichmann2001b,HS] I proved a generalization of the classical Frobenius Theorem on convenient manifolds, i.e. all manifolds which can appear in analysis under reasonable circumstances. The additional analytic requirement is surprisingly simple, namely the existence of a finite number of flows (or semiflows respectively).

THEOREM 1. *Let M be a convenient manifold and S a n-dimensional subbundle of TM. If the subbundle is involutive and for any point $m \in M$ there is an open neighborhood U and a local frame $\{A_i\}_{i=1,\ldots,n}$ such that A_i admit a local flow $Fl_t^{A_i}$ on U, then S is integrable, i.e. for any point $m \in M$ there is a unique maximal connected (integral) manifold $i : N_m \hookrightarrow M$ with immersion i and $T_x i(T_x N_m) = S_{i(x)}$ for $x \in N_m$. Furthermore we can construct the classical Frobenius chart.*

The notion of a Frobenius chart means that locally the integral submanifolds can be parallelized: they look like $\mathbb{R}^n$ in the model space up to diffeomorphisms. Furthermore the Theorem of Nelson as cited above is retreated in this new setting such that Nelson's condition can be replaced by a more analytic one (see [Teichmann2001b,HS] for details). The credo for this reasoning is that it is quite useful to analyse several analytic questions on "embedded" Fréchet spaces such as some closure of D with respect to a nice, countably defined topology, which was inspired by the work on my thesis.

EXAMPLE 2. *Assume in the above example that $D \subset H$ carries the initial Fréchet space topology with respect to the linear maps $A_1^{n_1} \cdots A_m^{n_m} : D \to H$ for $n_i \geq 0$, $i = 1, \ldots, d$ and assume that D is invariant with respect to the associated unital groups $S_t^j := \exp(tA_j)$, then we can construct a distribution $\mathcal{D}$ on the Fréchet space D spanned by the vector fields $A_1, \ldots, A_m$, which is involutive by condition 2.2. Therefore we can construct a foliation by Theorem 1 since the local frame $A_1, \ldots, A_m$ admits local (smooth) flows given by the restriction of S^j to D, which appear to be orbits of a Lie group action associated to the structure constants c_{ij}^k. This action can easily be extended to a strongly continuous unital Lie group action on H.*

Another key application is given by finite dimensional Realizations. Given a Banach space X, a generator A of a strongly continuous semigroup S and a locally Lipschitz map $P : \mathbb{R}_{\geq 0} \times U \subset \mathbb{R}_{\geq 0} \times X \to X$, we can consider the initial value problem

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + P(t, x(t)) \\ x(t_0) &= x_0 \in X, \end{aligned} \tag{2.3}$$

where we work with mild solutions, i.e. continuous maps which satisfy

$$x(t) = S(t)x_0 + \int_0^t S_{t-s}P(s, x(s))ds$$

for $t \in [0, \varepsilon[$. In the described case there exists a unique continuous solution for any initial value $x_0 \in U$ and any initial point in time $t_0 \geq 0$, see for example [FilipovicTeichmann2002b,HS].

One is interested in the obviously important question, whether such a solution can be represented by a finite dimensional differential equation in the following sense: there exists a map $\phi : U \subset \mathbb{R}^n_{\geq 0} \to X$ and a curve $y : [t_0, t_0 + \varepsilon[\to U$ such that $t \mapsto \phi(y(t))$ is a mild solution of 2.3. n, y and ϕ will depend on the "vector fields" $A + P_t$ and the initial value x_0. Notice that the $A + P_t$ is only densely defined, the vector field is continuous if and only if A is everywhere defined. The curve y will satisfy a differential equation, which can in principle be calculated from the vector fields $A + P_t$.

Provided that for all $x_0 \in \phi(U)$ we can find an appropriate curve y such that $t \mapsto \phi(y(t))$ solves 2.3 in the mild sense, we speak of a *finite dimensional realization (FDR)*. Needless to say that many of the classical, known solutions of non-linear PDEs are of this form.

The image $\phi(U)$, in case of an FDR, can be regarded as locally invariant submanifold with boundary of H (ignoring the possible singularities) with respect to the flow of mild solutions of equation 2.3. A submanifold with boundary $M \subset X$ is called *locally invariant with respect to* 2.3 if for $x_0 \in M$ and any starting point in time t_0 the unique solution of equation 2.3 stays on a small time interval in M. Therefore the existence of FDRs reduces to the question if we can find locally invariant submanifolds with boundary $M \subset H$, then take any chart to obtain nice curves y in the chart domain. One can formulate the following main theorem (see [FilipovicTeichmann2002c,HS]):

THEOREM 2. *Let H be Hilbert space and $M \subset H$ be a C^2-submanifold with boundary of H, then M is locally invariant with respect to 2.3 if and only if $\mathcal{M} \subset$*

$D(A)$ and

$$\Xi_t(h) := Ah + P(t,h) \in T_h\mathcal{M}$$

for all $h \in \mathcal{M}$ and $t \geq 0$, where $\Xi_t(h)$ is inward pointing for $h \in \partial\mathcal{M}$.

One should interpret this result also in the spirit of Dean Montgomery and Leon Zippin (see [MontgomeryZippin1957]): Invariance and Regularity are related. This led to the conclusions of the first part of [FilipovicTeichmann2002b,HS].

Now one is back to the problem of existence of manifolds tangent to some given "vector fields". From Frobenius Theory one knows that there is an algebraic obstruction to existence, which is not easy to formulate since the vector fields in question are neither everywhere defined nor smooth.

The main idea is to consider the problem on the natural Fréchet space $D(A^\infty) \subset H$ and to prove that under reasonable conditions on P this is sufficient. For this analysis assumptions on P have to be made, in particular $P : D(A^\infty) \to D(A^\infty)$ has to be well defined, but despite this we stay as general as possible. Additional questions can be solved by approximations.

The following Theorem, which is not yet published, is one cornerstone of this analysis - it generalizes the Frobenius Theorems given in [Teichmann2001b,HS] and [FilipovicTeichmann2002a,HS]. We ask under which conditions – given a set of vector fields $Y_1, \ldots, Y_n$ – a submanifold can be constructed at a point, such that the vector fields are tangent. I give here a condensed, rigorous proof for it.

The concepts of tangent vectors, tangent bundles, local flows and semiflows do not pose a problem (even for the semiflow case, where we need to differentiate on non-open domains) on convenient manifolds after the seminal work of Andreas Kriegl and Peter Michor (see [KrieglMichor1997] for all details). Given a convenient manifold M and smooth vector fields $Y_1, \ldots, Y_n$ on M. We consider two distributions on M, the first one, $\mathcal{D}$, is generated by vector fields $Y_1, \ldots, Y_n$ as in 2.1, and the second one, $\mathcal{D}_{LA}$, is generated by all iterated Lie brackets of the vector fields $Y_1, \ldots, Y_n$. We denote all linear combinations of these Lie brackets by locally defined smooth functions on M by $\mathfrak{E}$. In particular the evaluation of $\mathfrak{E}$ at a point $x \in M$ coincides with $\mathcal{D}_L A$.

A submanifold L is called a tangent submanifold for the distribution $\mathcal{D}$ if $\mathcal{D}(x) \subset T_x L$ for $x \in L$. A submanifold L is integral submanifold for the distribution $\mathcal{D}$ if $\mathcal{D}(x) = T_x L$ for $x \in L$.

Lemma 1. *Let $\mathcal{D}$ be a smooth distribution on M and L a tangent submanifold with inclusion $i : L \to M$, then for all $X \in C^\infty_{loc}(M \leftarrow \mathcal{D})$ the notion i^*X makes sense as an element of $C^\infty_{loc}(L \leftarrow TL)$, i.e. there is only one vector field on $i^{-1}(U) := L \cap U$ being i-related to X, where U is the domain of definition of X. Then the submanifold L is also tangent for the distribution $\mathcal{D}_{LA}$.*

Proof. Since i is an immersion we can define i^*X by Ti and we obtain smoothness by a submanifold chart. Pulling back elements $X, Y \in C^\infty(U \leftarrow \mathcal{D})$ we obtain

$$[X,Y] \circ i = Ti[i^*X, i^*Y]$$

by [KrieglMichor1997], in particular all iterated Lie brackets lie in $T_x i(T_x N)$ for $x \in N$, which is the very definition of being tangent. Therefore all iterated Lie brackets of $Y_1, \ldots, Y_d$ are tangent to N. □

Theorem 3. *Let $x \in M$ be given and assume that there are local vector fields $X_1, ..., X_n \in \mathfrak{E}$ around x, which are linearly independent at x and admit each a local* *flow, such that*

$$\mathcal{D}_{LA}(x) = \langle X_1(x), \dots, X_n(x) \rangle .$$

Then the following conditions are equivalent:

i) *There is finite dimensional integral submanifold L with $x \in L$ for the distribution $\mathcal{D}_{LA}$.*
ii) *There is an open convex subset $U \subset \mathbb{R}^n$ containing 0 such that*

$$\dim \mathcal{D}_{LA}(x) = \dim \mathcal{D}_{LA}(Fl_t^X \circ Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}(x))$$

for $(t, u_1, ..., u_n) \in U$, where $X = X_k$ for $k = 1, ..., n-1$.

Proof. Given an integral submanifold L with $x \in L$ for $\mathcal{D}_{LA}$, and a smooth vector field X taking values in $\mathcal{D}_{LA}$ and admitting a flow with domain containing x. Hence the flow lines of i^*X on N and of X on M coincide if the initial values coincide by uniqueness. Since $T_yL = \mathcal{D}_{LA}(y)$ for $y \in L$ the second condition is obvious since

$$Fl_t^X \circ Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}(x) \in N.$$

Take now $x \in M$ with $\dim \mathcal{D}_{LA}(x) = n \geq 1$ and the local vector fields $X_1, ..., X_n$ spanning $\mathcal{D}_{LA}$ at x. Given any $X := X_k$ for some $k = 1, ..., n-1$ and $y = Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}(x)$ for some fixed, small values $u_1, ..., u_n$, the vectors $X_1(Fl_t^X(y)), ..., X_n(Fl_t^X(y))$ span $\mathcal{D}_{LA}(Fl_t^X(y))$ for small t by the assumption, therefore there are smooth functions $f_{ij}(t)$ for small t with

$$[X_i, X](Fl_t^X(y)) = \sum_{j=1}^n f_{ij}(t) X_j(Fl_t^X(y))$$

by the assumption that $\mathfrak{E}$ is a Lie algebra. Consequently

$$\begin{aligned}
\frac{d}{dt}(Fl_t^X)^* X_i(y) &= [(Fl_t^X)^* X_i, X](y) = (Fl_t^X)^*[X_i, X](y) \\
&= T(Fl_{-t}^X)[X_i, X](Fl_t^X(y)) = \sum_{j=1}^n f_{ij}(t) T(Fl_{-t}^X) X_j(Fl_t^X(y)) \\
&= \sum_{j=1}^n f_{ij}(t)(Fl_t^X)^* X_j(y).
\end{aligned}$$

Hence the functions $g_i(t) := (Fl_t^X)^* X_i(y)$ satisfy the non-autonomous differential equation

$$\frac{d}{dt} g_i(t) = \sum_{j=1}^n f_{ij}(t) g_j(t)$$

with initial values $g_i(0) \in \mathcal{D}_{LA}(y)$. Consequently $g_i(t) \in \mathcal{D}_{LA}(y)$ for small t, so carefully written out

$$(Fl_t^{X_k})^* X_j(y) \in \mathcal{D}_{LA}(y)$$

for $j = 1, ..., n$ and $k = 1, ..., n-1$. The parametrization

$$\alpha(u_1, ..., u_n) := Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}(x)$$

defines therefore a submanifold with desired tangent spaces since

$$\frac{\partial}{\partial u_i}\alpha(u_1, ..., u_n) = (Fl_{u_1}^{X_1})_* \circ ... \circ (Fl_{u_i}^{X_i})_* X_i(\alpha(u_1, ..., u_n)) \in \mathcal{D}_{LA}(\alpha(u_1, ..., u_n)).$$

For details on the arguments see [Teichmann2001b,HS]. □

Remark 1. *If we restrict X_n in the above equivalence to admit only a local semiflow, then we get the analog equivalence for integral submanifolds with boundary. U then has to be open in $\mathbb{R}^n_{\geq 0}$. If we impose some real analyticity conditions one can drop condition ii.) by a variant of the Campbell-Baker-Hausdorff formula. Notice also that this solves the original problem whether we can construct a submanifold L tangent for $\mathcal{D}$ at $x \in M$. The minimal dimension of L is given by $\dim \mathcal{D}_L A$.*

The same reasoning can be applied to stochastic differential equations (having Stroock-Varadhan in mind). Nevertheless the situation is more ambitious since we are concerned with Hilbert space valued stochastic differential equations of the type

$$dr_t = (A + \alpha(r_t))dt + \sum_{i=1}^{d} \sigma_i(r_t) dB_t^i \qquad (2.4)$$

$$r_0 = r^* \in H$$

where $(B_t)_{t\geq 0}$ denotes a d-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ associated. Under appropriate Lipschitz conditions on $\alpha, \sigma_1, \ldots, \sigma_d : U \subset H \to H$ we obtain a local, mild solution theory, see [DaPratoZabczyk1992]. In particular $d + 1$ vector fields are involved here, which makes the analysis more ambitious.

Equation 2.4 is said to admit a *finite dimensional realization (FDR)* if, roughly speaking, there exists a map $\phi : U \subset \mathbb{R}^n_{\geq 0} \to H$ and for any $u \in U$ an n-dimensional diffusion state process $(Z_t)_{t\geq 0}$ exists such that $r_t = \phi(Z_t)$ solves equation 2.4 with initial value $\phi(u)$ in the mild sense. Notice that n, Z and ϕ depend on the initial points and the vector fields $A + \alpha, \sigma_1, \ldots, \sigma_d$. Again FDRs are essentially given by locally invariant finite dimensional submanifolds with boundary $\mathcal{M}$, i.e. for any $r_0 \in \mathcal{M}$ there is a strictly positive life time τ such that $r_t \in \mathcal{M}$ for $0 \leq t < \tau$, and we have the following Theorem.

Theorem 4. *Let H be a Hilbert space. Suppose that α is locally Lipschitz continuous and locally bounded, and σ is C^1. Let $\mathcal{M}$ be an n-dimensional submanifold with boundary of H. Then the following conditions are equivalent:*

i) *$\mathcal{M}$ is locally invariant for (2.4).*

ii) *$\mathcal{M} \subset D(A)$ and*

$$\Xi(h) := Ah + \alpha(h) - \frac{1}{2}\sum_{j=1}^{d} D\sigma_j(h)\sigma_j(h) \in T_h\mathcal{M}$$

$$\sigma_j(h) \in T_h\mathcal{M}, \quad j = 1, \ldots, d,$$

for all $h \in \mathcal{M}$, where $\Xi(h)$ is inward pointing and the $\sigma_j(h)$ are parallel to the boundary for $h \in \partial\mathcal{M}$.

After these preparations the following program can be set up for the treatment of equations like 2.4:

(P1): Formulate the problem of existence of locally invariant manifolds on $D(A^\infty)$, i.e. impose conditions on $\alpha, \sigma_1, \ldots, \sigma_d$.

(P2): Calculate the distributions $\mathcal{D}$ and $\mathcal{D}_{LA}$ and classify the cases when dimensions get finite.

(P3): Find either single locally invariant submanifolds, weak foliations or foliations and classify the leaves.

(P4): Analyse the question whether all locally invariant submanifolds are to be found in $D(A^\infty)$.

For the classification (P2) and the analysis of (P4) several methods (such as the notion of Banach maps) of the beautiful article [Hamilton1982] have been revisited, which makes the theory applicable since most of the existence assumptions can be dropped in this case and a full classification is possible. The framework of Banach map vector fields is flexible enough for applications and tractable enough for theoretic results.

(A1): $\sigma_j = \phi_j \circ \ell$ and $\alpha = \phi_0 \circ \ell$, where $\ell \in L(H, \mathbb{R}^p)$, for some $p \in \mathbb{N}$, and $\phi_j : \mathbb{R}^p \to D(A^\infty)$ are smooth for $j = 0, \dots, d$ and pointwise linearly independent maps for $1 \leq j \leq d$. Hence $\sigma : H \to D(A_0^\infty)^d$ is a Banach map.

(A2): For every $q \geq 0$, the linear map $(\ell, \ell \circ A, \dots, \ell \circ A^q) : D(A^\infty) \to \mathbb{R}^{p(q+1)}$ is open.

(A3): A is unbounded; that is, $D(A)$ is a strict subset of H. Equivalently, $A : D(A^\infty) \to D(A^\infty)$ is not a Banach map.

Assume (A1)–(A3). The above program (P1)–(P4) was worked out in [FilipovicTeichmann2002a,HS], namely (P1)–(P3) for the case of weak foliations, and in [FilipovicTeichmann2002b,HS], where we answered (P4). As a result it is sufficient to search in $D(A^\infty)$ for invariant submanifolds.

The geometry of the manifolds is a simple one, they are given by "bands" of subspaces of $D(A^\infty)$, which is referred to as affine structure in interest rate theory. It is remarkable that these are the only possible finite dimensional realizations.

2.2. The Term Structure of Interest Rates. The dynamics of Interest Rates is described by the Heath-Jarrow-Morton equation. Specifying a volatility structure, a market price of risk and an initial value determines the behaviour for all times. For certain volatility structures solution families are well known, which all together are finite dimensional realizations (as the Vasicek or Cox-Ingersoll-Ross models) of certain dimensions. It was one open problem if this collection of solutions is complete or if there are volatility structures admitting unknown finite dimensional realizations. A careful analysis of the generated distributions of vector spaces shows that the list of models can be viewed as complete and additionally this analysis provides a way to understand models from a more profound point of view. We also had the practitioner in mind, who is not satisfied with structure theory alone, but needs a full classification of all existing models with distinguishing properties such that she can choose an appropriate one. The analysis on these problems has been initiated by the seminal work of Tomas Björk and Lars Svensson in [BjörkSvensson1997].

A *term structure* is given by a point in a Hilbert space H of continuous curves, called forward rates. The arbitrate-free dynamics of the term structure is covered by Heath–Jarrow-Morton stochastic differential equation,

$$\begin{aligned} dr_t &= (A + \alpha_{HJM}(r_t))dt + \sum_{i=1}^{d} \sigma_i(r_t) dB_t^i \\ r_0 &= r^* \in H, \end{aligned}$$

where the field α_{HJM} can be calculated from the volatility vector fields $\sigma_1, \dots, \sigma_d$. The equation is written with respect to the risk neutral measure. There are three reasons why finite dimensional realizations are important in interest rate theory.

Consistency: A curve-fitting procedure should be consistent with an arbitrage-free stochastic model, that is, the model output curves should be of the curve-fitting type. Only such models can give a reasonable framework for the statistical comparison of the curve-fitting data over time.

Model calibration: Finite dimensional models with identifiable factors are inevitable for model calibration. Hence, given an arbitrary initial curve, the possible finite factor models evolving from this curve should be completely understood.

Analytical and computational tractability: For the purpose of calculating derivatives prices, the stochastic characteristics of the factor processes have to be known.

The following results could be achieved under the Banach map assumption on volatility vector fields. In applications volatility vector fields are typically of the form

$$\sigma_i(r) = \phi_i(l_1(r), ..., l_n(r))$$

with $\phi_i : U \subset \mathbb{R}^n \to H$ smooth and $l_k : H \to \mathbb{R}$ a continuous linear functional, which means that volatilities are sensitive to a finite number of traded bonds.

Geometry: The invariant submanifolds are seen to be of a simple affine form, which can be seen in the following parametrization,

$$\beta(u_0, u_1, \dots, u_n) = Fl_{u_0}(r^*) + \sum_{i=1}^{n} u_i \lambda^i,$$

where $\lambda^i \in D((\frac{d}{dx})^\infty)$ and $u_0 \mapsto Fl_{u_0}(r_*)$ is a Hilbert space valued smooth curve (*affine term structure).*

Factor process: The factor processes are seen to be *affine processes*, so particularly tractable Markov processes. In particular their generators are hypoelliptic and smooth densities exist. Here classical analysis in the spirit of Lars Hörmander was applied, see [Hörmander1983].

Coupling: If interest rate markets are coupled to stock markets and/or one wants to include stochastic volatilities, the above reasoning can be carried over on some extended phase space and similar results on affine term structures and affine processes can be proved.

This has been worked out in the review article [FilipovicTeichmann2002c,HS].

The analysis of long rates in this affine setting led to a general analysis of long interest rates, which was worked out in [HubalekKleinTeichmann2002].

2.3. Fundamentals of Lie Theory. My Thesis [Teichmann1999] treats the question, under which conditions an exponential map (or an evolution map) exists on infinite dimensional (convenient) Lie groups. The problem is a hard analytic problem, since one is concerned with the construction of solutions of right invariant differential equations on a given infinite dimensional Lie group, which are generically modeled on non-normable convenient vector spaces (so no existence Theorem for ordinary differential equations is at hand).

Here I developed the conviction that working on the convenient spaces at hand might be more useful than solving the problem on associated Sobolev hierarchies.

This approach was in a technically challenging way worked out by Hideki Omori et al. and led to the concept of strong ILB-groups.

I tried to find a characterization for the existence of evolution maps by Lipschitz metrics and the method of linearization, i.e. under the initial right regular representation of a Lie group G

$$\rho : G \to L(C^\infty(G, \mathbb{R})), g \mapsto (f \mapsto f(.g)).$$

Given a time dependent vector field $X \in C^\infty(\mathbb{R}, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of the Lie group G, an existing solution of the right invariant differential equation can be represented by product integrals in many cases. Taking this idea, I provide in [Teichmann2001d,HS] a characterization for the property, that product integral approximations stay (on compact time intervals) in compact sets – by Lipschitz metrics on the Lie groups. Lipschitz metric were proved to exist in [Teichmann2001d,HS] on all up to now known Lie groups.

In [Teichmann2001c,HS] I show – under some weak boundedness condition – that product integrals exist in convenient algebras as for example $L(C^\infty(G, \mathbb{R}))$. Applying this result of [Teichmann2001c,HS] finally leads to the existence of an evolution map by means of the linearization procedure and convenient calculus as developed in [KrieglMichor1997]: one takes product integral approximations, which stay in a compact set, maps this compact set via ρ to the convenient algebra $L(C^\infty(G, \mathbb{R}))$, continues by proving convergence of product integrals, since the boundedness condition is satisfied, and concludes for the preimage the desired convergence by initiality. At the beginning of this research the description of regular abelian Lie groups was considered, see [MichorTeichmann1998].

EXAMPLE 3. *Let G be a nice convenient Lie group and $c : \mathbb{R} \to G$ a smooth curve with $c(0) = e$. By Lipschitz metrics we can conclude that the set $\{c(\frac{t}{n})^n\}_{n\geq 1}$ is sequentially compact on compact time intervals. The representation ρ maps this set to the compact, hence bounded set*

$$\{(\rho \circ c)(\frac{t}{n})^n\}_{n\geq 1}.$$

So we are in the assumption of [Teichmann2001c,HS] *and are able conclude that $\lim_{n\to\infty}(\rho \circ c)(\frac{t}{n})^n$ is a smooth group in the convenient algebra $L(C^\infty(G, \mathbb{R}))$. By compactness we know the $c(\frac{t}{n})^n$ has adherence points for any t, which by continuity of ρ are mapped to $(\rho \circ c)(\frac{t}{n})^n$. But ρ is also injective, so the adherence points are unique and we can conclude that there is a curve $t \mapsto T_t$ in G such that*

$$(\rho \circ T)(t) = \{(\rho \circ c)(\frac{t}{n})^n\}_{n\geq 1}.$$

This then leads also to smoothness of T by Frölicher space properties and the fact that ρ is initial. Hence $\lim_{n\to\infty} c(\frac{t}{n})^n = T_t$ and T is a smooth one-parameter subgroup. A fortiori $T_t = \exp(tc'(0))$ for $t \geq 0$. This way an existence Theorem for exponential maps can be proved.

The work on regularity was inspiring for further research on infinite dimensional Lie groups: In [HallerTeichmannVizman2002,HS] we determine the Riemannian manifolds for which the group of exact volume preserving diffeomorphisms is a totally geodesic subgroup of the group of volume preserving diffeomorphisms, considering right invariant L^2-metrics. The same is done for the subgroup of Hamiltonian diffeomorphisms as a subgroup of the group of symplectic diffeomorphisms. These are special cases of totally geodesic subgroups of diffeomorphisms with Lie algebras big enough to detect the vanishing of a symmetric 2-tensor field.

In [HallerTeichmann2002] we find on a closed smooth manifold M equipped with k fiber bundle structures whose vertical distributions span the tangent bundle, that every smooth diffeomorphism f of M sufficiently close to the identity can be written as a product $f = f_1 ... f_n$, where f_i preserves the i^{th}-fiber. The factors f_i can be chosen smoothly in f. We apply this result to show that on a certain class of closed smooth manifolds every diffeomorphism sufficiently close to the identity can be written as product of commutators and the factors can be chosen smoothly. Furthermore we get concrete estimates on how many commutators are necessary. For the proof we apply the hard inverse function theorem of Nash-Moser.

2.4. Semigroups on convenient vector spaces. The work on semigroups consists (such as classical semigroup theory) of a Hille-Yosida result relating resolvent properties to the existence of a semigroup and Trotter-type approximation results. In [Teichmann2001c,HS] an existence result for initial value problems on convenient vector spaces is proved by Trotter approximations.

DEFINITION 1 (Product integral). *Let A be a convenient algebra. Given a smooth curve $X\colon \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R}^2 \to A$ with $h(s,0) = e$ and $\frac{\partial}{\partial t} h(s,0) = X(s)$, then we define the following finite products of smooth curves*

$$p_n(a,t,h) := \prod_{i=0}^{n-1} h(a + \frac{(n-i)(t-a)}{n}, \frac{t-a}{n})$$

for $a, t \in \mathbb{R}$. If p_n converges in all derivatives to a smooth curve $c : \mathbb{R} \to A$, then c is called the product integral *of X or h and we write $c(a,t) = \prod_a^t \exp(X(s)ds)$ or $c(a,t) =: \prod_a^t h(s,ds)$. The case $h(s,t) = c(t)$ with $p_n(0,t,h) = c(\frac{t}{n})^n$ is referred to as* simple product integral.

THEOREM 5 (Approximation theorem). *Let A be convenient algebra. Given a smooth curve $X : \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R}^3 \to A$ with $h(u,r,0) = e$ and $\frac{\partial}{\partial t} h(u,r,0) = X_u(r)$. Suppose that for every fixed $s_0 \in \mathbb{R}$, there is $t_0 > s_0$ such that $p_n(u,s,t,h)$ is bounded in A on compact (u,s,t)-sets and for all $n \geq 1$. Then the product integral $\prod_s^t h(u,r,dr)$ exists and the convergence is Mackey in all derivatives on compact (u,s,t)-sets. Furthermore the product integral is the right evolution of X_u, i.e.*

$$\frac{\partial}{\partial t} \prod_s^t h(u,r,dr) = X_u(t) \frac{\partial}{\partial t} \prod_s^t h(u,r,dr),$$
$$\prod_s^s h(u,r,dr) = e.$$

This result applies nicely on spaces, where the functional analytic structure is not well understood as $L(C^\infty(G,\mathbb{R}))$. The forthcoming article [Teichmann2001a,HS], which was a sidestep in the work of [Teichmann2001c,HS], treats the Hille-Yosida-Theorem on convenient vector spaces. Due to the large class of spaces one can restrict the analysis to a smaller class of semigroups, namely smooth semigroups, for which a nice version of the Hille-Yosida-Theorem can be proved.

3. Perspectives

In **Differential Geometry** I got particularly interested in the following open questions of research:

- **(Q1):** Analysis of finite dimensional realizations for (well-known) non-linear PDEs.
- **(Q2):** Analysis of geodesic equations on infinite dimensional Lie groups under the perspective of FDRs.
- **(Q3):** Analysis of geodesic equations under the developed Lipschitz approximation procedures.

In **Interest Rate Theory (Financial Mathematics) and Stochastic Analysis** the following problems are focused in future research:

- **(Q4):** Analysis of singular FDRs, even though their importance is restricted in financial mathematics.
- **(Q5):** Justification of the concept of FDRs for concrete applications. If the generated distribution $\mathcal{D}_{LA}$ is infinite dimensional the solutions should have particular (statistical) features, which could be described more precisely. This also leads into the beautiful world of stochastic differential geometry in the spirit of Paul Malliavin or Daniel Stroock.
- **(Q6):** Revisiting the theory of stochastic filters by the new methods on Fréchet spaces.
- **(Q7):** Analysis of other term structure problems such as future markets in the challenging direction of energy markets (derivation and foundation of new models).

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