ON MARTIN HAIRER'S THEORY OF REGULARITY STRUCTURES

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1. Introduction

Martin Hairer received the Fields medal at the ICM in Seoul 2014 for "his outstanding contributions to the theory of stochastic partial differential equations" (quoted from the ICM webpage), in particular for the creation of the theory of regularity structures. Martin was born 1975 into an Austrian family living in Switzerland: his father, Ernst Hairer, is a well-known mathematician in numerical analysis working at the University of Geneva. Martin's mother has worked as a teacher in elementary school and in a Ludothek, his sister works in medical management, and his brother teaches sports. Martin completed his PhD at Geneva university under the supervision of Jean-Pierre Eckmann in 2001. He is Regius Professor of mathematics at the University of Warwick, having previously held a position at the Courant Institute of New York University. He is married to the mathematician Xue-Mei Li, who also works at University of Warwick. Martin develops quite successfully the audio editor software "Amadeus", which silently reveals the Austrian background.

Martin Hairer's work on the solution of the KPZ equation and on regularity structures is astonishing by its self-contained character, its crystal-clear exposition and its far-reaching conclusions. I have rarely read research papers where a new theory is built in a such convincing and lucid way.

The purpose of this article is to explain some elements of Martin Hairer's work on regularity structures and some aspects of my personal view on it. I am not able to appreciate or even to describe the history, meaning and value of all problems on stochastic partial differential equations, which can be solved with the theory of regularity structures, but I do believe that there are still many future applications, for instance in Mathematical Finance or Economics, to come.

2. Systems and noise

Loosely speaking there are two reasons to include random influences into deterministic descriptions of a system's time evolution: either there are absolute sources of noise related to fundamental laws of physics, which need to be considered for a full description of a system, or there are subjective sources of noise due to a fundamental lack of information, which in turn can be modeled as random influences on the system. In both cases the irregularity of the concrete noise can lead to quite ill-posed equations even though it is often clear, for instance through numerical

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experiments, that the corresponding equations are reasonable. This can already be seen when simple stochastic systems like

$$\frac{d}{dt}S_t = S_t\dot{W}_t \, , \; S_0 > 0$$

are considered. This is a linear growth equation, where the growth rate is white noise $\dot{W}_t$, e.g. (independent) Gaussian shocks with vanishing expectation and covariance $\delta(t-s)$. Simulation of this equation is simple but to understand the formula analytically already needs Ito's theory. Even though white noise $\dot{W}_t$ is only defined in the distributional sense, already its integral with respect to time is a (Hölder) continuous function, which gives hope for the previous equation to have a reasonable interpretation, if one is able to define integrals along Brownian motion.

There are two main approaches towards such noisy systems: deterministic approaches which consider any realization of noise as an additional deterministic input into the system, or stochastic approaches which consider a realization of a noise as stochastic input into the system. Rough path theory and regularity structures belong to the first approach, stochastic analysis constitutes the second one.

The problem with (white) noises – by its very nature – is its persistent irregularity or roughness. Let us consider again the simplest example of noise: white noise, or its integrated version, Brownian motion. Taking a more physical point of view white noise models velocity of a Brownian particle

$$W_t := \int_0^t \dot{W}_s ds \, ,$$

i.e. a particle moving on continuous trajectories with independent increments being random variables with vanishing expectation and variance proportional to time. Simulation is easy, but apparently Brownian motion due to the independent nature of its increments is a complicated mathematical object whose existence is already an involved mathematical theorem. Brownian motion is on the other hand an extremely important tool for modeling random phenomena. For instance many important models in Mathematical Finance, where independent Gaussian shocks are a modeling assumption, are driven by Brownian motions, for instance the Black-Merton-Scholes model, the Heston model, the SABR model, etc.

Let us consider, e.g., the length of a Brownian trajectory to understand one crucial aspect of irregularity. Consider first quadratic variation which is approximated by

$$\sum_{i=0}^{2^n} \left(W_{ti/2^n} - W_{t(i-1)/2^n}\right)^2 .$$

By arguments going back to Paul Lévy it is clear that the previous sum converges almost surely to t, which means in turn that almost surely Brownian motion has infinite length since any continuous curve with finite length would have a vanishing quadratic variation. Therefore naive Lebesgue-Stieltjes integration with respect to Brownian motion is not an option, also Brownian trajectories are almost surely too rough for Young integration.

Mathematically speaking one needs an integration theory with respect to curves, which are not of finite total variation (like Brownian motion), in order to make sense of equations involving Brownian motion or white noise (like the Black-Merton-Scholes equation). Kiyoshi Ito's approach to deal with this problem is stochastic integration: by arguments from L^2-martingale theory a particular set of integrands, namely locally square integrable predictable ones, is singled out such that limits of Riemannian sums exist almost surely. Ito's insight allows, e.g., to solve stochastic differential equation in its integral form by fixed point arguments. For instance the Black-Merton-Scholes equation now reads

$$S_t = S_0 + \int_0^t S_s dW_s \, ,$$

where the right hand side is defined via stochastic integration and well understood in its probabilistic and analytic properties. Ito's stochastic calculus is a wonderful tool to work with, in particular in Mathematical Finance, as long as stochastic integration works well. This, however, might get problematic if one considers more general stochastic processes, which are not real-valued (as in the Black-Scholes equation) anymore but, e.g., distribution-valued like multi-dimensional white noises.

Brownian motion appears as integral of a one-dimensional white noise, but we also can consider multivariate versions of white noises, where independent shocks in a space-time manner appear. Of course the irregularity of such noises is worse. The analogue of W_t, i.e. integration with respect to time, is not function valued anymore but only defined in the distributional sense. Hence non-linear equations containing space-time white noises need to come up with a theory how to define non-linear functions of generalized functions, which is a well-known and hard problem. Such multivariate white noises appear in several important equations from physics, and even in equations of Mathematical Finance, it is important to understand non-linear equations where such noises appear. Regularity structures provide a way to solve this problem in a surprisingly elegant way.

3. Regularity structures

Regularity structures have been introduced by Martin Hairer in a series of papers to provide solution concepts for Stochastic partial differential equations (SPDEs) like the Kardar-Parisi-Zhang (KPZ) equation, the Φ^4_3 equation, the parabolic Anderson model, etc. These equations often came so far with excellent motivations from mathematical physics, convincing solution chunks in several regimes and surprisingly deep conclusions from those, but rarely with mathematically satisfying (dynamic) solution concepts. For a discussion of these issues we refer mainly to Martin Hairer's article [Hairer(2014a)] in Inventiones Mathematicae, but also to the great introductory paper [Hairer(2014b)].

Let us take for instance the Φ^4_3-equation

$$\partial_t u = \Delta u - u^3 + \xi$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}^3$ (with periodic boundary conditions in space), where u is a scalar function and ξ is a space-time white noise. This important SPDE from quantum mechanics has a very singular additive term, namely space-time white noise. If one counts (parabolically) time twice as much as space dimension, ξ is regular of order $-\frac{5}{2} - \epsilon$, for any $\epsilon > 0$ (we denote this by $-\frac{5}{2}^-$). This irregularity cannot be regularized by the heat kernel, which raises regularity only by 2 (due to well-known Schauder estimates). The resulting object is not a function yet and therefore any non-linear operation on it is problematic. If, however, we consider the mild formulation of the Φ^4_3-equation through convolution with the Green's function K

$$u = K * (\xi - u^3) + K * u_0 \ ,$$

and if we hope for a Banach fixed point argument, we are faced with non-linear operations on $K * \xi$ already after one iteration step: in particular we need an interpretation of $(K * \xi)^3$. It has been shown by Martin Hairer that under suitable, now well understood re-normalizations of the Φ^4_3 equation of the type

$$\partial_t u_\epsilon = \Delta u_\epsilon - u^3 + C_\epsilon u_\epsilon + \xi_\epsilon \ ,$$

where $\xi_\epsilon \to \xi$ as $\epsilon \to 0$ is a mollification of the white noise and $C_\epsilon \to \infty$ is some constant depending on the mollification, one can formulate a solution concept: indeed the limit $u_\epsilon \to u$ as $\epsilon \to 0$ exists, u does not depend on the particular mollification involved, and the structure of the "infinities" is actually well described. More details can be found in Section 6 of [Hairer(2014b)] or Section 9 of [Hairer(2014a)]. The solution theory of the dynamic Φ^4_3 is one convincing argument for the power and beauty of regularity structures.

The theory of regularity structures is based on a natural still ingenious split between algebraic properties of an equation and the analytic interpretation of those algebraic structures. These considerations are profoundly motivated by renormalization theory from mathematical physics, however, the crucial point is their precise mathematical meaning and their clear structure. Another source of inspiration for regularity structures is Terry Lyons' Rough Path Theory, see [Lyons(2006)], which is somehow extended from curves to functions on higher dimensional spaces, an aspect which is outlined in the book [FrizHairer(2014)].

The bridge between the algebraic and analytic world is done by the so called reconstruction operator $\mathcal{R}$, whose existence is a beautiful result from wavelet analysis interesting by itself. Regularity structures also come with precise numerical approximation results and are therefore very useful to establish numerical techniques. In the sequel we highlight on the cornerstones of the regularity structures without any proofs but with emphasis on meaning and ideas. Proofs are analytically involved, but due to the intriguing structure of the theory, which is briefly highlighted in the sequel, all the hours spent with [Hairer(2014a)] fly by quickly.

Let $A \subset \mathbb{R}$ be an index set, bounded from below and without accumulation point, and let $T = \bigoplus_{\alpha \in A} T_\alpha$ be a direct sum of Banach spaces T_α graded by A. Let furthermore G be a group of linear operators on T such that, for every $\alpha \in A$, every $\Gamma \in G$, and every $\tau \in T_\alpha$, one has $\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta$.

The triple $\mathcal{T} = (A, T, G)$ is called a *regularity structure* with *model space* T and *structure group* G. Given $\tau \in T$, we shall write $\|\tau\|_\alpha$ for the norm of its T_α-projection.

What about the meaning of elements of T: they represent expansions of "functions" at some space-time point in terms of "model functions" of regularity α, namely elements of T_α. In order to make this precise Martin Hairer introduces "models", which do nothing else than mapping an abstract expansion to a generalized function (respecting Hölder regularity orders) for each point in space time.

Let us be a bit more precise on that in the sequel: given a test function ϕ on $\mathbb{R}^d$, we write ϕ^λ_x as a shorthand for the re-scaled function

$$\phi^\lambda_x(y) = \lambda^{-d}\phi\big(\lambda^{-1}(y - x)\big) \ .$$

For $r > 0$ we denote by B_r the set of all functions $\phi : \mathbb{R}^d \to \mathbb{R}$ with $\phi \in C^r$, its norm $\|\phi\|_{C^r} \leq 1$ and supported in the unit ball around the origin. At this point we can say what we mean by *regularity of order* $\alpha < 0$ of a distribution η, namely that there exists a constant C such that the inequality

$$\big|\eta(\phi_x^\lambda)\big| \leq C\lambda^\alpha$$

holds uniformly over $\phi \in B_r$, $x \in K$ and $\lambda \in]0,1]$. Regularity of order $\alpha \geq 0$ will just be Hölder regularity.

Given a regularity structure $\mathcal{T}$ and an integer $d \geq 1$, a *model* for $\mathcal{T}$ on $\mathbb{R}^d$ consists of maps

$$\begin{aligned} \Pi \colon \mathbb{R}^d &\to L\big(T, \mathcal{D}'(\mathbb{R}^d)\big) \qquad & \Gamma \colon \mathbb{R}^d \times \mathbb{R}^d &\to G \\ x &\mapsto \Pi_x & (x,y) &\mapsto \Gamma_{xy} \end{aligned}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x \Gamma_{xy} = \Pi_y$. Furthermore, given $r > |\inf A|$, for any compact set $K \subset \mathbb{R}^d$ and constant $\gamma > 0$, there exists a constant C such that the inequalities

$$\big|\big(\Pi_x \tau\big)(\phi_x^\lambda)\big| \leq C\lambda^\alpha \|\tau\|_\alpha \;, \qquad \|\Gamma_{xy}\tau\|_\beta \leq C|x-y|^{\alpha-\beta}\|\tau\|_\alpha \;,$$

hold uniformly over $\phi \in B_r$, $(x,y) \in K$, $\lambda \in]0,1]$, $\tau \in T_\alpha$ with $\alpha \leq \gamma$, and $\beta < \alpha$. In words: for every space-time point $x \in \mathbb{R}^d$ the distribution $\Pi_x\tau$ interprets each $\tau \in T$ accordingly. The role of the group G also becomes clear at this point: G is a collection of linear maps on T which encode how expansions of a fixed analytic object transform when considering different space-time points.

Models interpret abstract expansions (elements of T) at each space time point x: these model functions constitute a frame (at each point in space time) on which one can construct generalized functions expressible in this frame with point-varying coordinates. Martin Hairer calls these generalized functions "modeled distributions". In particular they depend on the model (Π, Γ).

Given a regularity structure $\mathcal{T}$ equipped with a model (Π, Γ) over $\mathbb{R}^d$, the space $\mathcal{D}^\gamma = \mathcal{D}^\gamma(\mathcal{T}, \Gamma)$ is given by the set of functions $f : \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$ such that, for every compact set K and every $\alpha < \gamma$, the exists a constant C with

$$\|f(x) - \Gamma_{xy}f(y)\|_\alpha \leq C|x-y|^{\gamma-\alpha} \tag{3.1}$$

uniformly over $x, y \in K$.

A priori it is not at all clear whether modeled distributions actually allow for an interpretation as distribution on space time: the most fundamental result in the theory of regularity structures then states that given a modeled distribution $f \in \mathcal{D}^\gamma$ with $\gamma > 0$, there exists a *unique* distribution $\mathcal{R}f$ on $\mathbb{R}^d$ such that, for every $x \in \mathbb{R}^d$, $\mathcal{R}f$ equals $\Pi_x f(x)$ near x up to order γ. More precisely, one has the following reconstruction theorem, whose proof relies on deep results from wavelet analysis (for the beautiful proof see Martin Hairer's Inventiones article [Hairer(2014a)], Theorem 3.10):

Let $\mathcal{T}$ be a regularity structure and let (Π, Γ) be a model for $\mathcal{T}$ on $\mathbb{R}^d$. Then, there exists a unique linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}'(\mathbb{R}^d)$ such that

$$\big|\big(\mathcal{R}f - \Pi_x f(x)\big)(\phi_x^\lambda)\big| \lesssim \lambda^\gamma \;, \tag{3.2}$$

uniformly over $\phi \in B_r$ and $\lambda \in]0,1]$, and locally uniformly in $x \in \mathbb{R}^d$.

In other words: modeled distributions, i.e. functions into the space of abstract expansions, come with appropriate consistency conditions (3.1), which still allow to construct a distribution such that the model (Π, Γ) remains tangent up to given order γ as in (3.2). Notice that the existence of the reconstruction operator should be interpreted as the construction of an integral

$$(\mathcal{R}f)(\phi) = \text{“} \int \sum_{\alpha \in A} f_\alpha(x)(\Pi_x \tau_\alpha)(x)\phi(x)dx \text{”} \ ,$$

where we write $f(x) = \sum_{\alpha \in A} f_\alpha(x)\tau_\alpha$ in a sloppy way for $f \in \mathcal{D}^\gamma$ and where we write a value for the distribution $\Pi_x \tau_\alpha$ at x even though this is not possible in all cases of interest. Notice also that the existence of such integrals is highly non-trivial, since we are performing summations over very singular objects.

At this point it might become clear where regularity structures are leading us: translate a “real world equation” into an equation on abstract expansions, solve this equation and translate the solution – via the reconstruction operator back to “real world”. To realize this idea it is necessary to understand how linear equations are translated to abstract expansion spaces, which is the world of *Schauder estimates*. Classical Schauder estimates tell that for a kernel $K : \mathbb{R}^d \to \mathbb{R}$ being smooth everywhere, except for a singularity at the origin of approximate homogeneity $\beta - d$ for some $\beta > 0$, the integral operator $K : f \mapsto K * f$ maps $\mathcal{C}^\alpha$, i.e. α-Hölder continuous functions, into $\mathcal{C}^{\alpha+\beta}$ for every $\alpha \in \mathbb{R}$ (except for those values for which $\alpha + \beta \in \mathbb{N}$). In the theory of regularity structures this naturally amounts to lifting the integral operator K to an operator on modeled distributions $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$, which commutes with reconstruction, i.e. $K * \mathcal{R}f = \mathcal{R}\mathcal{K}f$ for all modeled distributions $f \in \mathcal{D}^\gamma$.

Martin Hairer needs three basic ingredients for such a type of construction:

(1) in order to describe the behavior of regular (smooth in the classical sense) parts of the integral operator K polynomials should be part of the given regularity structure.
(2) an abstract integration operator $\mathcal{I} : T \to T$, which encodes the action of K on singular objects.
(3) a compatibility of the given model, abstract integration and the to-be-lifted integral kernel K, i.e. between $\Pi_x \mathcal{I}\tau$ and $K * \Pi_x \tau$ near x.

This leads to the introduction of admissible models where the desired lift $\mathcal{K}$ can actually be performed for certain kernels of homogeneity $\beta - d$. All proofs can be found in Section 5 of [Hairer(2014a)].

With all these ingredients we can return to the Φ^4_3 equation introduced at the beginning of this section and see how regularity structures enter the stage: we can construct a tailor-made regularity structure, which is generated in a free way by variables representing ξ, $(K * \xi)^3$, $K * (K * \xi)^3$, etc, and by space-time polynomials. Let us replace K by an abstract integration operator $\mathcal{I}$, ξ by an abstract variable Ξ of order $-\frac{5}{2}^-$, and u by Φ, then we can – having Banach's fixed point theorem in mind – build a regularity structure generated by Ξ, $\mathcal{I}(\Xi)^3$, etc. Associated orders of regularity are $-\frac{5}{2}^-$, $-\frac{3}{2}^-$, etc. There will be only finitely many generators of negative orders in a minimal model space T (i.e. a Banach fixed point consideration makes sense). The Φ^4_3-equation is then translated into a fixed point equation on the coefficient space $\mathcal{D}^\gamma$ with respect to this roughly described regularity structure $\mathcal{T}$, i.e.

$$\Phi = \mathcal{I}(\Xi - \Phi^3) + \text{ polynomials } ,$$

where 'polynomials' denote terms describing the 'smooth' part of the operator $\mathcal{K}$.

Apparently the equation can be solved abstractly by the very construction in some $\mathcal{D}^\gamma$ spaces if we consider models where Ξ is mapped to a mollified noise, since then it is easy to define the meaning of objects like $\mathcal{I}(\Xi)^3$. Reconstruction then yields a solution of the Φ^4_3-equation with mollified noise and initial value u_0.

If, however, the mollified noise converges to white noise, the actual power of the regularity structures is revealed: as outlined we obtain solutions for each model (Π, Γ), first in $\mathcal{D}^\gamma$, and second – via reconstruction – as distributions on space time if noise is mollified. However, the algebraic structure of T prescribes precisely which "products" of distributions have to be defined appropriately, which in turn means to construct models even for singular noises. This procedure involves re-normalizations, i.e. the real world equation being satisfied after reconstruction changes. Notice that re-normalizing products has to be done only for finitely many elements of T of regularity order less than zero (in this sub-critical situation) to guarantee a well-defined model, well-defined modeled distributions and well-defined reconstruction. Re-normalization groups, whose dimension depends on the particular situation, govern the structure of different models which can be defined as limits of models with mollified noises. Hence the re-normalization of the real world equation is transferred to re-normalization of models, which can be analyzed by methods from group theory and algebra. The corresponding solution concept of the SPDE does not depend on the chosen mollification and satisfies all necessary requirements, hence deserves to be called *the* solution of the Φ^4_3-equation.

4. Ito calculus, rough paths and Regularity Structures

Let us consider two natural examples of regularity structures, which explain the following discussion:

4.1. The polynomial regularity structure. The most classical example of a regularity structure is the polynomial one: the abstract expansion are abstract polynomials in d variables $X^1, \dots, X^d$, models are concrete polynomials, $A = \mathbb{N}$ and we can identify G with the group of translations acting on $\mathbb{R}[X^1, \dots, X^d]$ via $\Gamma_h p := p(. - h)$ for $h \in \mathbb{R}^d$.

The canonical polynomial model is then given by

$$\big(\Pi_x X^k\big)(y) = (y-x)^k \ , \qquad \Gamma_{xy} = \Gamma_{y-x} \ . \tag{4.1}$$

If we choose the canonical polynomial model, then the space of modeled distributions $\mathcal{D}^\gamma$ corresponds to the space of Hölder continuous functions $\mathcal{C}^\gamma$ (with proper understanding for integer γ). In other words the canonical regularity structure on $\mathbb{R}^d$ speaks about Hölder functions and their local Taylor expansions (see Section 3 of [Hairer(2014b)]).

4.2. A regularity structure for rough paths. Given $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $n > 1$. Define $A = \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$. We consider a free vector space T generated by n order α elements W_j, n order $\alpha - 1$ elements Ξ_j, and n^2 order $2\alpha - 1$ elements $W_j \Xi_i$, and one order 0 element 1.

We choose $G = \mathbb{R}^n$ and define the action on T via

$$\Gamma_x 1 = 1 \ , \qquad \Gamma_x \Xi_i = \Xi_i \ , \qquad \Gamma_x W_i = W_i - x^i \ , \qquad \Gamma_x (W_j \Xi_i) = W_j \Xi_i - x^j \Xi_i \ .$$

This is a regularity structure on $\mathbb{R}$ and models for this regularity structure are precisely rough paths of order α with values in $\mathbb{R}^n$. Let us be more precise on this. Take a model (Π, Γ), then the formulas

$$\big(\Pi_s 1\big)(t) = 1 \, , \qquad \big(\Pi_s W_j\big)(t) = X^j_t - X^j_s$$

$$\big(\Pi_s \Xi_j\big)(\psi) = \int \psi(t)\, dX^j_t \, , \qquad \big(\Pi_s W_j \Xi_i\big)(\psi) = \int \psi(t)\, d\mathbb{X}^{i,j}_{s,t} \, ,$$

for test functions ψ define a (geometric) rough path $(s,t) \mapsto (X^j_t - X^j_s, \mathbb{X}^{i,j}_{s,t})$, furthermore $\Gamma_{su} = \Gamma_{X_u - X_s}$. Apparently X only needs to be Hölder continuous of order α, whereas $\mathbb{X}$ satisfies a sort of Hölder condition of order 2α. The algebraic relationships for models translate to the relationships for rough paths.

Modeled distributions appear in this setting as natural integrands with respect to rough paths: take $Y \Xi_j + Y'_i W_i \Xi_j \in \mathcal{D}^{3\alpha - 1}$, then reconstruction actually defines a curve, which can be seen as the integral $\int Y dX_j$ (see Section 3 of [Hairer(2014b)]).

At this point it is clear that the split between algebraic structures, which depend on the concrete SPDE, and their analytic reconstruction is crucial for the flexibility on introducing re-normalizations. In contrast Ito's stochastic calculus does work differently: stochastic integration is introduced via martingale arguments, so the problem that the increment of Brownian motion on an interval of length Δt only scales like $\sqrt{\Delta t}$ is cured by predictability of integrands. No additional term of order Δt is introduced to replace the missing order. More precisely: locally in time the stochastic integral $\int_0^t h_s dW_s$ looks like $h_s \Delta W_s$, hence miraculously Riemannian sums converge even though the local expansion is only given up to order $\sqrt{\Delta t}$.

Rough path theory, such as regularity theory, argues that an integral along a path with low regularity, like a trajectory of Brownian motion, can only be defined if an additional term of order Δt is introduced describing the integral locally up to order Δt. The set of integrands changes with respect to Ito integration: for stochastic integration every locally square integrable predictable integrand is eligible, whereas in case of rough path theory predictability is replaced by several analytic conditions (in particular also anticipative integrands are possible).

Ito's approach has the advantage of a robust large set of integrands, namely all bounded predictable processes, which work for a large set of integrators, namely all semi-martingales. Not only Brownian motion is a possible integrator but also Lévy processes, or more general jump processes. On the other hand regularity of the stochastic integral is low: usually a stochastic integral depends only in a measurable way on the integrator and not better.

Rough path theory or the theory of regularity structures in contrast has the advantage of a considerably more regular dependence of the integral on the integrator. The price to pay is a less robust, more regular set of integrands. To be precise here, the regularity is described in terms of regularities of the reconstruction operator, which depends in a continuous way on the model. This amazing fact means for instance in case of the SABR model (a particular stochastic volatility model)

$$dX_t = X_t Y_t dW^1_t \, , \; dY_t = \sigma Y_t dW^2_t \, , \; X_0 \geq 0 \, , \; Y_0 \geq 0 \, ,$$

that the solution process (X,Y) only depends in a measurable way on the Brownian input paths (W^1, W^2), whereas in a continuous way on Brownian motion together with its Lévy areas $(W^1, W^2, \int W^1 dW^2 - W^2 dW^1)$ (which is an essential part of a model describing integration with respect to Brownian motion in the world of regularity structures). In other words: the expected higher regularity of the solution map of a stochastic differential equation is discovered by introducing stochastic integration in a deterministic way. This important insight is the content of Terry Lyons' Universal Limit Theorem of rough path theory, which appears now as one particular case of a regularity structure, see for instance [Lyons(2006)]

It remains one interesting topic for future research to describe settings where both somehow complementary approaches to deal with rough objects, namely stochastic integration and the theory of regularity structures, are combined.

References

[FrizHairer(2014)] P. Friz, M. Hairer. A course on rough paths. Springer 2014.

[Hairer(2014a)] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[Hairer(2014b)] M. Hairer. Introduction to Regularity Structures. to appear in Braz. Jour. Prob. Stat., 2014.

[Lyons(2006)] T. Lyons. Differential equations driven by rough paths. Lecture notes from St. Flour, 2006.

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