

LECTURE NOTES ON THE YAMADA–WATANABE CONDITION FOR THE PATHWISE UNIQUENESS OF SOLUTIONS OF CERTAIN STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In these lecture notes we discuss the Yamada–Watanabe condition for the pathwise uniqueness of the solution of certain stochastic differential equations. This condition is weaker than the usual Lipschitz condition, the proof is based on Bihari’s inequality. An important application in mathematical finance is the Cox–Ingersoll–Ross model.

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1. INTRODUCTION AND BASIC DEFINITIONS

These lecture notes explain the Yamada–Watanabe condition, which relaxes the Lipschitz condition for the pathwise uniqueness of solutions of stochastic differential equations (SDEs) of the type

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \quad t \geq 0.$$

Hence, this condition can be used to show the strong uniqueness of solutions of a SDE with certain non-Lipschitz coefficients. The main references for this part are [2], [3], [5] and [6].

In mathematical finance, this is of particular interest for the Cox–Ingersoll–Ross model (CIR model for short), which describes the stochastic evolution of interest rates $(r_t)_{t \geq 0}$ by the SDE

$$dr_t = \alpha(\mu - r_t) dt + \sigma\sqrt{r_t} dW_t, \quad t \geq 0,$$

with $r_0 \geq 0$, where α, μ with $\alpha\mu \geq 0$ and σ denote real constants.

Before stating the main theorem, we start with some definitions necessary for the sequel.

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Definition 1.1. Given two jointly Borel measurable functions $b: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$, a solution of the stochastic differential equation,

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \quad t \geq 0, \quad (1.2)$$

is a pair (X, W) of continuous adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and such that

- (a) $W = (W_t)_{t \geq 0}$ is a standard $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion with values in \mathbb{R}^d ,
- (b) the integrals implicitly given by (1.2) are well defined, i.e., for all $t \geq 0$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$,

$$\int_0^t \sigma_{ij}^2(s, X_s) ds \stackrel{\text{a.s.}}{<} \infty \quad \text{and} \quad \int_0^t |b_i(s, X_s)| ds \stackrel{\text{a.s.}}{<} \infty,$$

- (c) for all $i \in \{1, 2, \dots, n\}$, the component processes of $X = (X_1, \dots, X_n)^\top$ satisfy, up to indistinguishability,

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}(s, X_s) dW_s^j, \quad t \geq 0. \quad (1.3)$$

Definition 1.4. We say that there is pathwise uniqueness for the SDE (1.2) if whenever (X, W) and (\tilde{X}, \tilde{W}) are two solutions of (1.2) defined on the same filtered probability space with $W = \tilde{W}$ (same Brownian motion) and $X_0 \stackrel{\text{a.s.}}{=} \tilde{X}_0$ (same \mathcal{F}_0 -measurable initial condition), then X and \tilde{X} are indistinguishable, that is, there exists a set $N \in \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ with $\mathbb{P}[N] = 0$ such that $\{X_t \neq \tilde{X}_t\} \subset N$ for all $t \in [0, \infty)$.

2. YAMADA–WATANABE CONDITION FOR SDEs

In this section, we give the main result, which combines the one- and multi-dimensional case. However, we mention that the Yamada–Watanabe condition is essentially a one-dimensional result (see Remarks 2 and 3 in [6]).

In the following, we use $|\cdot|$ for the n -dimensional Euclidean norm, and $\|\cdot\|$ for the Frobenius matrix norm, that is $\|A\| = \sqrt{\text{tr}[AA^\top]}$, where $\text{tr}[AA^\top]$ denotes the trace of a matrix AA^\top . For the proof we assume that the filtration is right-continuous.

Theorem 2.1. *Consider the stochastic differential equation (1.2). Assume that there exist functions $\kappa, \varrho: [0, \infty) \rightarrow [0, \infty)$ satisfying $\kappa(0) = 0$,*

$$|b(t, x) - b(t, y)| \leq \kappa(|x - y|), \quad (2.2)$$

and

$$\|\sigma(t, x) - \sigma(t, y)\| \leq \varrho(|x - y|) \quad (2.3)$$

for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^n$. Furthermore, assume that ϱ is non-decreasing, $\varrho(u) > 0$ for all $u > 0$ and

$$\int_0^\varepsilon \frac{du}{\varrho^2(u)} = \infty \quad \forall \varepsilon > 0, \quad (2.4)$$

In addition, assume that there exists a non-decreasing, concave and continuous function $G: [0, \infty) \rightarrow [0, \infty)$ with $G(0) = 0$, strictly positive for all $u > 0$, such that

$$G(u) \geq \kappa(u) + \frac{n-1}{2u} \varrho^2(u) \quad \forall u > 0 \quad (2.5)$$

and

$$\int_0^\varepsilon \frac{du}{G(u)} = \infty \quad \forall \varepsilon > 0. \quad (2.6)$$

Then the pathwise uniqueness of solutions of (1.2) holds.

Remark 2.7. Note that for $n = 1$ with the choice $G = \kappa$, the conditions on G are actually conditions on κ , in particular (2.6) reduces to

$$\int_0^\varepsilon \frac{du}{\kappa(u)} = \infty \quad \forall \varepsilon > 0. \quad (2.8)$$

For $n \geq 2$ and vanishing drift b , we can choose κ to be the zero function. With the choice $G(u) = \frac{n-1}{2u} \varrho^2(u)$ for $u > 0$, the condition (2.6) is equivalent to

$$\int_0^\varepsilon \frac{u}{\varrho^2(u)} du = \infty \quad \forall \varepsilon > 0, \quad (2.9)$$

which is substantially more restrictive than (2.4).

Example 2.10. For the continuous concave function κ satisfying (2.8), typical examples are $\kappa(u) = Cu$ with a constant $C > 0$ as well as

$$\kappa(u) = \begin{cases} 0 & \text{for } u = 0, \\ Cu \log \frac{1}{u} & \text{for } u \in (0, e^{-2}), \\ C(u + e^{-2}) & \text{for } u \geq e^{-2}, \end{cases} \quad (2.11)$$

which has unbounded slope close to the origin. The example (2.11) satisfies (2.8), because for $0 < \delta \leq \varepsilon < 1$

$$\int_\delta^\varepsilon \frac{du}{u \log \frac{1}{u}} = \log \log \frac{1}{\delta} - \log \log \frac{1}{\varepsilon} \rightarrow \infty \quad \text{as } \delta \searrow 0.$$

For the function ϱ satisfying (2.4), one can always take $\varrho(u) = \sqrt{\kappa(u)}$ for $u \geq 0$, where κ satisfies (2.8), and also $\varrho(u) = Cu^\alpha$ with exponent $\alpha \geq 1/2$ is possible. If κ satisfies (2.8), then $\varrho(u) := \sqrt{u\kappa(u)}$ for $u \geq 0$ satisfies (2.9) and $G(u) := \kappa(u) + \frac{n-1}{2u} \varrho^2(u)$ for $u \geq 0$ satisfies (2.6) for every $n \in \mathbb{N}$. Note that the Lipschitz-continuous case corresponding to $\varrho(u) = \kappa(u) = Cu$ is included.

Remark 2.12 (Deterministic dynamics). By setting the diffusion function σ equal to zero and $\varrho(u) = \sqrt{u\kappa(u)}$ for $u \in [0, \infty)$, Theorem 2.1 also weakens the Lipschitz condition for the uniqueness of solutions of ordinary initial value problems to (2.8). Note that the function κ (or G , respectively) doesn't need to be concave in this case, because the application of Jensen's inequality for (2.26) is not necessary (and the entire proof simplifies substantially).

Corollary 2.13 (Application to the CIR model). *In one dimension, if the drift b is Lipschitz continuous, and σ is Hölder continuous with exponent $1/2$, i.e., there exists some constant $C > 0$ such that*

$$|\sigma(t, x) - \sigma(t, y)| \leq C\sqrt{|x - y|} \quad \forall 0 \leq t < \infty \text{ and } x, y \in \mathbb{R},$$

then the pathwise uniqueness for the solution of equation (1.2) holds.

Proof of Theorem 2.1. The main idea of the proof is to construct a sequence $(f_k)_{k \in \mathbb{N}}$ of C^2 -functions $f_k: \mathbb{R}^n \rightarrow [0, \infty)$ approximating the Euclidean norm $\mathbb{R}^n \ni z \mapsto |z|$, such that Itô's multi-dimensional formula can be applied to $f_k(X_t - Y_t)$, where X and Y are two solutions of the SDE (1.2) with $X_0 \stackrel{\text{a.s.}}{=} Y_0$. By then passing to the

limit $k \rightarrow \infty$, the aim is to show that $\mathbb{E}[|X_t - Y_t|] = 0$ for all $t \geq 0$, which implies pathwise uniqueness.

The first step is to construct such approximations. Due to assumption (2.4), there exists a sequence

$$1 = a_0 > a_1 > a_2 > \cdots > a_k \searrow 0$$

such that

$$\int_{a_k}^{a_{k-1}} \frac{du}{\varrho^2(u)} = k, \quad k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$, we can construct a continuous function $\phi_k: [0, \infty) \rightarrow [0, \infty)$ such that

$$\phi_k(u) \begin{cases} \leq \frac{2}{k\varrho^2(u)} & \text{for } u \in (a_k, a_{k-1}), \\ = 0 & \text{otherwise,} \end{cases} \quad (2.14)$$

and

$$\int_{a_k}^{a_{k-1}} \phi_k(u) du = 1,$$

because the upper bound (2.14) of ϕ_k integrates to 2 over (a_k, a_{k-1}) . Next we define the auxiliary function $\varphi_k: [0, \infty) \rightarrow [0, \infty)$ by

$$\varphi_k(w) = \int_0^w \int_0^v \phi_k(u) du dv, \quad w \geq 0.$$

Note that φ_k is a twice continuously differentiable function with $\varphi_k(w) = 0$ for $w \in [0, a_k]$. Furthermore, $\varphi_k'(w) = \int_0^w \phi_k(u) du$ for $w \geq 0$, hence

$$\varphi_k'(w) \begin{cases} = 0 & \text{for } w \in [0, a_k], \\ \leq 1 & \text{for } w \in (a_k, a_{k-1}), \\ = 1 & \text{for } w \in [a_{k-1}, \infty). \end{cases} \quad (2.15)$$

Therefore, the sequence $(\varphi_k)_{k \in \mathbb{N}}$ is monotone increasing with $w - a_{k-1} \leq \varphi_k(w) \leq w$ for all $w \in [a_{k-1}, \infty)$. Finally, we can define the approximating sequence $(f_k)_{k \in \mathbb{N}}$ by

$$f_k(z) := \varphi_k(|z|), \quad k \in \mathbb{N}, z \in \mathbb{R}^n.$$

It follows that each f_k is a continuously differentiable function on \mathbb{R}^n and that $f_k(z) \nearrow |z|$ uniformly in $z \in \mathbb{R}^n$ as $k \rightarrow \infty$.

Now, let X and Y be two solutions of (1.2) with $X_0 \stackrel{\text{a.s.}}{=} Y_0$, driven by the same d -dimensional Brownian motion, and define the difference process Z by

$$Z_t := X_t - Y_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \quad t \geq 0,$$

where we simplified the notation by defining the \mathbb{R}^n -valued stochastic process

$$b_s = b(s, X_s) - b(s, Y_s), \quad s \geq 0,$$

and the matrix-valued stochastic process

$$\sigma_s = \sigma(s, X_s) - \sigma(s, Y_s), \quad s \geq 0.$$

By assumption (2.2),

$$|b_s| \leq \kappa(|Z_s|), \quad s \geq 0. \quad (2.16)$$

We note that the definition of the Frobenius matrix norm and assumption (2.3) imply

$$\text{tr}[\sigma_s \sigma_s^\top] = \|\sigma_s\|^2 \leq \varrho^2(|X_s - Y_s|) = \varrho^2(|Z_s|), \quad s \geq 0. \quad (2.17)$$

Fix $k \in \mathbb{N}$. Applying Itô's multi-dimensional formula to $f_k(Z_t)$, we obtain up to indistinguishability,

$$f_k(Z_t) = I_k(t) + J_k(t), \quad t \geq 0, \quad (2.18)$$

with

$$I_k(t) := \int_0^t \nabla f_k(Z_s) \sigma_s dW_s, \quad t \geq 0, \quad (2.19)$$

and

$$J_k(t) := \int_0^t \left(\nabla f_k(Z_s) b_s + \frac{1}{2} \operatorname{tr}[H_k(Z_s) \sigma_s \sigma_s^\top] \right) ds, \quad t \geq 0, \quad (2.20)$$

where $\nabla f_k(z)$ and $H_k(z)$ denote the gradient vector and the Hessian matrix of f_k at $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, respectively. We will now fix $t \geq 0$ and define suitable stopping times, so that we can treat the expectation of these two terms.

The stochastic process I_k is a local martingale starting at zero, hence there exists an increasing sequence $(T_{k,l})_{l \in \mathbb{N}}$ of stopping times with $T_{k,l} \rightarrow \infty$ as $l \rightarrow \infty$ such that, for every $l \in \mathbb{N}$, the process $M_{k,l}(s) := I_k(s \wedge T_{k,l})$ with $s \geq 0$ is a uniformly integrable martingale. For every $m \in \mathbb{N}$ define $U_m = \inf\{s \geq 0 : |Z_s| > m\}$. Since the filtration is assumed to be right-continuous, every U_m is a stopping time. Since Z has continuous paths, $U_m \rightarrow \infty$ as $m \rightarrow \infty$. By Doob's optional stopping theorem,

$$\mathbb{E}[I_k(t \wedge T_{k,l} \wedge U_m)] = 0, \quad l, m \in \mathbb{N}. \quad (2.21)$$

Note that $\nabla f_k(z) = 0$ for $|z| \leq a_k$ and $\nabla f_k(z) = \varphi'_k(|z|)z/|z|$ for $|z| \geq a_k$, hence $|\nabla f_k(z)| \leq 1$ for all $z \in \mathbb{R}^n$ by (2.15). Therefore, by the Cauchy–Schwarz inequality and estimate (2.16),

$$|\nabla f_k(Z_s) b_s| \leq |\nabla f_k(Z_s)| |b_s| \leq \kappa(|Z_s|), \quad s \geq 0. \quad (2.22)$$

The diagonal components of the Hessian matrix are given by

$$(H_k(z))_{i,i} = \frac{\partial^2 f_k(z)}{\partial z_i^2} = \phi_k(|z|) \frac{z_i^2}{|z|^2} + \varphi'_k(|z|) \frac{|z|^2 - z_i^2}{|z|^3}, \quad i \in \{1, \dots, n\},$$

(remember that φ'_k and ϕ_k are zero in a neighborhood of the origin). Since φ'_k is uniformly bounded by one, see (2.15), the above equation implies that

$$\operatorname{tr}[H_k(z)] = \phi_k(|z|) + \varphi'_k(|z|) \frac{n-1}{|z|} \leq \phi_k(|z|) + \frac{n-1}{|z|} \mathbb{I}_{\{z \neq 0\}}, \quad z \in \mathbb{R}^n. \quad (2.23)$$

Note that φ_k is convex on $[0, \infty)$ by construction, also the Euclidean norm is convex on \mathbb{R}^n , hence f_k is convex. Therefore, the Hessian H_k of f_k is positive semi-definite everywhere. Since also $\sigma_s \sigma_s^\top$ is positive semi-definite, it follows that

$$\operatorname{tr}[H_k(Z_s) \sigma_s \sigma_s^\top] \leq \operatorname{tr}[H_k(Z_s)] \operatorname{tr}[\sigma_s \sigma_s^\top], \quad s \geq 0. \quad (2.24)$$

Combining this inequality with (2.17) and (2.23) in the first step and using (2.14) in the second one implies that

$$\begin{aligned} \operatorname{tr}[H_k(Z_s) \sigma_s \sigma_s^\top] &\leq \phi_k(|Z_s|) \varrho^2(|Z_s|) + \frac{n-1}{|Z_s|} \varrho^2(|Z_s|) \mathbb{I}_{\{Z_s \neq 0\}} \\ &\leq \frac{2}{k} + \frac{n-1}{|Z_s|} \varrho^2(|Z_s|) \mathbb{I}_{\{Z_s \neq 0\}}, \quad s \geq 0. \end{aligned}$$

Using (2.22) and the bound (2.5) for G , we obtain for the integrand in (2.20) that

$$|\nabla f_k(Z_s) b_s| + \frac{1}{2} \operatorname{tr}[H_k(Z_s) \sigma_s \sigma_s^\top] \leq \frac{1}{k} + G(|Z_s|), \quad s \geq 0. \quad (2.25)$$

It follows from (2.18) that, for all $l, m \in \mathbb{N}$,

$$\mathbb{E}[f_k(Z_{t \wedge T_{k,l} \wedge U_m})] = \mathbb{E}[I_k(t \wedge T_{k,l} \wedge U_m)] + \mathbb{E}[J_k(t \wedge T_{k,l} \wedge U_m)].$$

The first expectation on the right-hand side vanishes due to (2.21). Combining (2.20) with (2.25) and noting that G is non-negative, it follows that

$$\mathbb{E}[f_k(Z_{t \wedge T_{k,l} \wedge U_m})] \leq \frac{t}{k} + \int_0^t \mathbb{E}[G(|Z_{s \wedge U_m}|)] ds \quad l, m \in \mathbb{N}.$$

Since by assumption G is concave on $[0, \infty)$, Jensen's inequality implies that

$$\mathbb{E}[G(|Z_{s \wedge U_m}|)] \leq G(\mathbb{E}[|Z_{s \wedge U_m}|]), \quad m \in \mathbb{N}, s \geq 0. \quad (2.26)$$

Letting $l \rightarrow \infty$, using Fatou's lemma, it follows that

$$\mathbb{E}[f_k(Z_{t \wedge U_m})] \leq \frac{t}{k} + \int_0^t G(\mathbb{E}[|Z_{s \wedge U_m}|]) ds, \quad m \in \mathbb{N}.$$

Letting $k \rightarrow \infty$ and using monotone convergence theorem, we obtain for the difference process Z the estimate

$$\mathbb{E}[|Z_{t \wedge U_m}|] \leq \int_0^t G(\mathbb{E}[|Z_{s \wedge U_m}|]) ds, \quad m \in \mathbb{N}, t \geq 0. \quad (2.27)$$

The stopping at U_m makes sure that $[0, \infty) \ni s \mapsto \mathbb{E}[|Z_{s \wedge U_m}|]$ is real-valued and continuous (apply the dominated convergence theorem). Due to (2.6) and Bihari's inequality, see Theorem 3.1(b), estimate (2.27) implies that $\mathbb{E}[|Z_{t \wedge U_m}|] = 0$ for all $m \in \mathbb{N}$. Since $U_m \rightarrow \infty$ as $m \rightarrow \infty$, an application of Fatou's lemma shows that $\mathbb{E}[|Z_t|] = 0$, hence $X_t \stackrel{\text{a.s.}}{=} Y_t$. Since this holds for all rational $t \geq 0$ and since the processes X and Y have continuous paths they are indistinguishable. \square

3. BIHARI'S INEQUALITY

Bihari's inequality [1, 4], proved by Hungarian mathematician Imre Bihari (1915–1998), is a nonlinear generalization of the Grönwall–Bellman inequality¹. It is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equations.

Theorem 3.1. *Let I denote an interval of the real line of the form $[a, \infty)$, $[a, b]$ or $[a, b)$ with $a < b$. Let $\beta, u: I \rightarrow [0, \infty)$ and $w: [0, \infty) \rightarrow [0, \infty)$ be three continuous functions, where w is non-decreasing and $w(y) > 0$ for all $y \in [0, \infty)$.*

(a) *If, for some $\alpha > 0$, the function u satisfies the inequality*

$$u(t) \leq \alpha + \int_a^t \beta(s) w(u(s)) ds, \quad t \in I, \quad (3.2)$$

then

$$u(t) \leq F^{-1}\left(\int_a^t \beta(s) ds\right), \quad t \in [a, T), \quad (3.3)$$

where F^{-1} is the inverse function of

$$F(x) := \int_\alpha^x \frac{dy}{w(y)}, \quad x > 0,$$

¹See also *Grönwall's inequality* at en.wikipedia.org/wiki/, version of October 28, 2010.

and

$$T := \sup \left\{ t \in I \mid \int_a^t \beta(s) ds < \int_\alpha^\infty \frac{dy}{w(y)} \right\}.$$

(b) If the function u satisfies the inequality (3.2) with $\alpha = 0$ and

$$\int_0^x \frac{dy}{w(y)} = \infty \quad \text{for all } x > 0, \quad (3.4)$$

then $u(t) = 0$ for all $t \in I$.

Remark 3.5. If $\int_\alpha^\infty \frac{dy}{w(y)} = \infty$, then (3.3) is valid on $[0, \infty)$. An example of such a function is $w(y) = y$ for $y \in [0, \infty)$.

Proof of Theorem 3.1. (a) Denoting the right-hand side of (3.2) by

$$v(t) := \alpha + \int_a^t \beta(s) w(u(s)) ds, \quad t \in I,$$

we have $u \leq v$ on I by (3.2), which implies that $w(u(s)) \leq w(v(s))$ for all $s \in I$ since w is non-decreasing. Using the definitions of F and v as well as this inequality, it follows that

$$\frac{dF(v(s))}{ds} = \frac{v'(s)}{w(v(s))} = \frac{\beta(s) w(u(s))}{w(v(s))} \leq \beta(s), \quad s \in I.$$

Integrating this between a and t and using $F(v(a)) = F(\alpha) = 0$,

$$F(v(t)) = F(v(t)) - F(v(a)) \leq \int_a^t \beta(s) ds, \quad t \in I.$$

Since F is strictly increasing,

$$v(t) \leq F^{-1} \left(\int_a^t \beta(s) ds \right), \quad t \in [a, T).$$

Since $u(t) \leq v(t)$, the inequality (3.3) follows.

(b) Consider any $t \in I$ and $x > 0$. Due to (3.4) there exists $\alpha \in (0, x]$ such that

$$\int_\alpha^x \frac{dy}{w(y)} = \int_a^t \beta(s) ds.$$

Since u also satisfies (3.2) with this α , (3.3) implies that

$$u(t) \leq F^{-1} \left(\int_a^t \beta(s) ds \right) = x.$$

Since $x > 0$ was arbitrary, $u(t) = 0$ follows. \square

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