A Generalization of Panjer's Recursion and Numerically Stable Risk Aggregation

Workshop on Credit Risk
Université d'Evry Val d'Essonne, June 25–27, 2008

Uwe Schmock
(Joint work with S. Gerhold and R. Warnung)
Christian Doppler Laboratory
for Portfolio Risk Management (PRisMa Lab)
Financial and Actuarial Mathematics
Vienna University of Technology, Austria
www.fam.tuwien.ac.at

Workshop on Portfolio Risk Management

Mo., Sept. 29, 2008, Vienna, Austria

Organized by

Research Group for
Financial and Actuarial Mathematics and
CD-Laboratory for Portfolio Risk Management
Institute for Mathematical Methods in Economics
Vienna University of Technology
A-1040 Vienna, Austria

www.fam.tuwien.ac.at/prisma2008

Motivation: The Collective Model

Task: Calculate the distribution of the random sum

$$S = X_1 + \cdots + X_N$$

of N losses, where the loss sizes $\{X_i\}_{i\in\mathbb{N}}$ are i. i. d. and independent of N.

Applications:

- Claims in a homogeneous insurance portfolio
- Losses in a credit portfolio ($\rightarrow$ extended CreditRisk$^+$)
- Operational losses (Basel II), aggregation for every line of business and loss type.

Standard tool: Panjer's recursion for specific distributions of N, when the X_i are $\mathbb{N}_0$-valued.

Loss Number Distributions in the Panjer Class

Definition: A probability distribution $\{q_n\}_{n\in\mathbb{N}_0}$ is said to belong to the Panjer(a, b, k) class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \cdots = q_{k-1} = 0$ and

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1} \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq k+1.$$

Important examples: (all distributions are known)

- Poisson$(\lambda) \in$ Panjer$(0, \lambda, 0)$ with intensity $\lambda > 0$
- NegBin$(\alpha, p) \in$ Panjer$(q, (\alpha-1)q, 0)$ with $\alpha > 0$, probability $p \in (0, 1)$ and $q := 1 - p$
- Log$(q) \in$ Panjer$(q, -q, 1)$ with $q \in (0, 1)$ and $q_n = -\frac{q^n}{n \log(1-q)}$ for all $n \in \mathbb{N}$

Extended Logarithmic Distribution

For $k \in \mathbb{N}\backslash\{1\}$ and $q \in (0,1]$ define $q_0 = \cdots = q_{k-1} = 0$,

$$q_n = \frac{\binom{n}{k}^{-1} q^n}{\sum_{l=k}^{\infty} \binom{l}{k}^{-1} q^l} \quad \text{for } n \geq k.$$

ExtLog(k, q) is in the Panjer$(q, -kq, k)$ class.

Extended Negative Binomial Distribution

For $k \in \mathbb{N}$, $\alpha \in (-k, -k+1)$ and $p \in [0,1)$ define $q = 1 - p$, $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{\alpha+n-1}{n} q^n}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } n \geq k.$$

ExtNegBin(α, k, p) is in the Panjer$(q, (\alpha-1)q, k)$ class.

Extended Panjer Recursion

If $\mathcal{L}(N) \in$ Panjer(a, b, k), independent of the i. i. d. $\mathbb{N}_0$-valued sequence $\{X_n\}_{n\in\mathbb{N}}$, and $a\mathbb{P}(X_1 = 0) \neq 1$, then $S := X_1 + \cdots + X_N$ satisfies

$$\mathbb{P}(S = 0) = \varphi_N\big(\mathbb{P}(X_1 = 0)\big)$$

with φ_N the probability generating function of N, and

$$\mathbb{P}(S = n) = \frac{1}{1 - a\mathbb{P}(X_1 = 0)} \bigg(\mathbb{P}(S_k = n)\mathbb{P}(N = k) + \sum_{j=1}^{n} \Big(a + \frac{bj}{n}\Big) \mathbb{P}(X_1 = j)\mathbb{P}(S = n - j) \bigg)$$

for all $n \in \mathbb{N}$, where $S_k = X_1 + \cdots + X_k$.

Example for Numerical Instability

Take $N \sim \text{ExtNegBin}(\alpha, k, p)$ with $k \in \mathbb{N}$, $\varepsilon, p \in (0,1)$ and $\alpha := -k + \varepsilon$. Consider the loss distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = l) = 1/2$ with $l \geq 3$. Then

$$p_{k+l} = \mathbb{P}(S = k+l) = q\frac{k(l-1)+\varepsilon k}{k+l}\Big(\frac{q_k}{2^{k+1}} + \frac{q_{k+l-1}}{k2^{k+l}}\Big) - q\frac{k(l-1)-\varepsilon l}{k+l}\frac{q_k}{2^{k+1}}.$$

With $\varepsilon = 1/10\,000$, $k = 1$, $l = 5$, $p = 1/10$:

$$p_6 = 0.1499\,926 - 0.1499\,701 = 0.0000\,225\,.$$

Panjer recursion with five significant digits gives

$$p_6 = 0.0000\,400\ldots \qquad (\approx 78\% \text{ relative error}).$$

Panjer Recursion Replaced by Weighted Convolution

Fix $l \in \mathbb{N}$, consider $N \sim \{q_n\}_{n\in\mathbb{N}_0}$ and $\tilde{N}_i \sim \{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$, define $S = X_1 + \cdots + X_N \sim \{p_n\}_{n\in\mathbb{N}_0}$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i} \sim \{\tilde{p}_{i,n}\}_{n\in\mathbb{N}_0}$ for $i \in \{1,\ldots,l\}$. Assume there exist $k \in \mathbb{N}_0$ and $a_1,\ldots,a_l,b_1,\ldots,b_l \in \mathbb{R}$ such that

$$q_n = \sum_{i=1}^{l}\Big(a_i + \frac{b_i}{n}\Big)\tilde{q}_{i,n-i} \quad \text{for } n \geq k+l$$

and $\tilde{q}_{i,0} = \cdots = \tilde{q}_{i,k+l-i-1} = 0$ for $i \in \{1,\ldots,l\}$.
Then $p_0 = \varphi_N\big(\mathbb{P}(X_1 = 0)\big)$ and, for $n \in \mathbb{N}$,

$$p_n = \sum_{j=1}^{k+l-1}\mathbb{P}(S_j = n)q_j + \sum_{i=1}^{l}\sum_{j=0}^{n}\Big(a_i + \frac{b_i j}{in}\Big)\mathbb{P}(S_i = j)\tilde{p}_{i,n-j}.$$

Combination of Truncated Distributions

Fix $k \in \mathbb{N}_0$, $l \in \mathbb{N}$. For all $i \in \{1,\dots,l\}$ assume that $\alpha_i \geq 0$, $\beta_i \geq -i\alpha_i$ (at least one $\neq$) and that the $\mathbb{N}_0$-valued $\tilde{N}_i$ satisfies $\mathbb{P}(\tilde{N}_i < k+l-i) = 0$. Consider $q_0,\dots,q_{k+l-1} \geq 0$ with $q_0 + \cdots + q_{k+l-1} \leq 1$. Define

$$q_n = c\sum_{i=1}^{l}\Bigl(\alpha_i + \frac{\beta_i}{n}\Bigr)\mathbb{P}(\tilde{N}_i = n-i) \quad \text{for } n \geq k+l,$$

$$c = \biggl(1 - \sum_{n=0}^{k+l-1} q_n\biggr)\bigg/\sum_{i=1}^{l}\biggl(\alpha_i + \beta_i\,\mathbb{E}\biggl[\frac{1}{i+\tilde{N}_i}\biggr]\biggr).$$

Then $\{q_n\}_{n\in\mathbb{N}_0}$ is a probability distribution satisfying the recursion condition with $a_i = c\alpha_i$ and $b_i = c\beta_i$ and the calculation of $\{p_n\}_{n\in\mathbb{N}_0}$ is numerically stable.

Weighted Convolution for ExtLog

Let $k \in \mathbb{N}$ and $q \in (0,1)$. Let $\tilde{N} \sim \text{ExtLog}(k,q)$ and $N \sim \text{ExtLog}(k+1,q)$, where ExtLog$(1,q)$ means Log(q). Define $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$ and $S = X_1 + \cdots + X_N$. Then, with an explicit $b_1 > 0$, the weighted convolution

$$\mathbb{P}(S=n) = \frac{b_1}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=j)\,\mathbb{P}(\tilde{S}=n-j), \quad n \in \mathbb{N},$$

is numerically stable. Numerically stable algorithm:

- Panjer recursion for Log(q)
- $k-1$ weighted convolutions: Log$(q) \to$ ExtLog$(2,q)$ $\to \cdots \to$ ExtLog$(k-1,q) \to$ ExtLog(k,q)

Numerically Stable Algorithm for ExtLog(2,1)

Let $N \sim \text{ExtLog}(2,1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S=0) = \mathbb{P}(X_1=0) + \mathbb{P}(X_1 \geq 1)\log \mathbb{P}(X_1 \geq 1)$$

with $0 \log 0 := 0$ and, in the case $\mathbb{P}(X_1 \geq 1) > 0$,

$$\mathbb{P}(S=n) = \frac{1}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=j)\,r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = -\log \mathbb{P}(X_1 \geq 1)$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \geq 1)}\biggl(\mathbb{P}(X_1=n) + \frac{1}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=n-j)\,r_j\biggr)$$

Weighted Convolution for ExtNegBin

Let $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ and $p \in (0,1)$. Let $\tilde{N} \sim \text{ExtNegBin}(\alpha, k, p)$ and $N \sim \text{ExtNegBin}(\alpha-1, k+1, p)$, where $\text{ExtNegBin}(\alpha, 0, p)$ means $\text{NegBin}(\alpha, p)$. Define $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$ and $S = X_1 + \cdots + X_N$. Then

$$\mathbb{P}(S=n) = \frac{b_1}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=j)\,\mathbb{P}(\tilde{S}=n-j), \quad n \in \mathbb{N},$$

with an explicit $b_1 > 0$ is numerically stable. Algorithm:

- Panjer recursion for $\text{NegBin}(\alpha+k, p)$
- k weighted convolutions:
  $\text{NegBin}(\alpha+k, p) \to \text{ExtNegBin}(\alpha+k-1, 1, p) \to \cdots$
  $\to \text{ExtNegBin}(\alpha+1, k-1, p) \to \text{ExtNegBin}(\alpha, k, p)$

Stable Algorithm for ExtNegBin($\alpha - 1, 1, 0$)

Let $N \sim \text{ExtNegBin}(\alpha - 1, 1, 0)$ with $\alpha \in (0, 1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S = 0) = 1 - \big(\mathbb{P}(X_1 \geq 1)\big)^{1-\alpha}$$

and in the case $\mathbb{P}(X_1 \geq 1) > 0$

$$\mathbb{P}(S = n) = \frac{1 - \alpha}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j)\,r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = \big(\mathbb{P}(X_1 \geq 1)\big)^{-\alpha}$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \geq 1)} \sum_{j=1}^{n} \frac{n - j + \alpha j}{n} \mathbb{P}(X_1 = j)\,r_{n-j}\,.$$

Tempered α-Stable Distributions on $[0, \infty)$

Let Y be α-stable on $[0, \infty)$ with Laplace transform

$$\mathcal{L}_Y(s) = \mathbb{E}[\exp(-sY)] = \exp(-\gamma_{\alpha,\sigma} s^{\alpha}), \quad s \geq 0,$$

where $\alpha \in (0, 1)$, $\sigma > 0$ and $\gamma_{\alpha,\sigma} := \sigma^{\alpha} / \cos\big(\frac{\alpha\pi}{2}\big)$.
For $\tau \geq 0$ define τ-tempered α-stable distribution

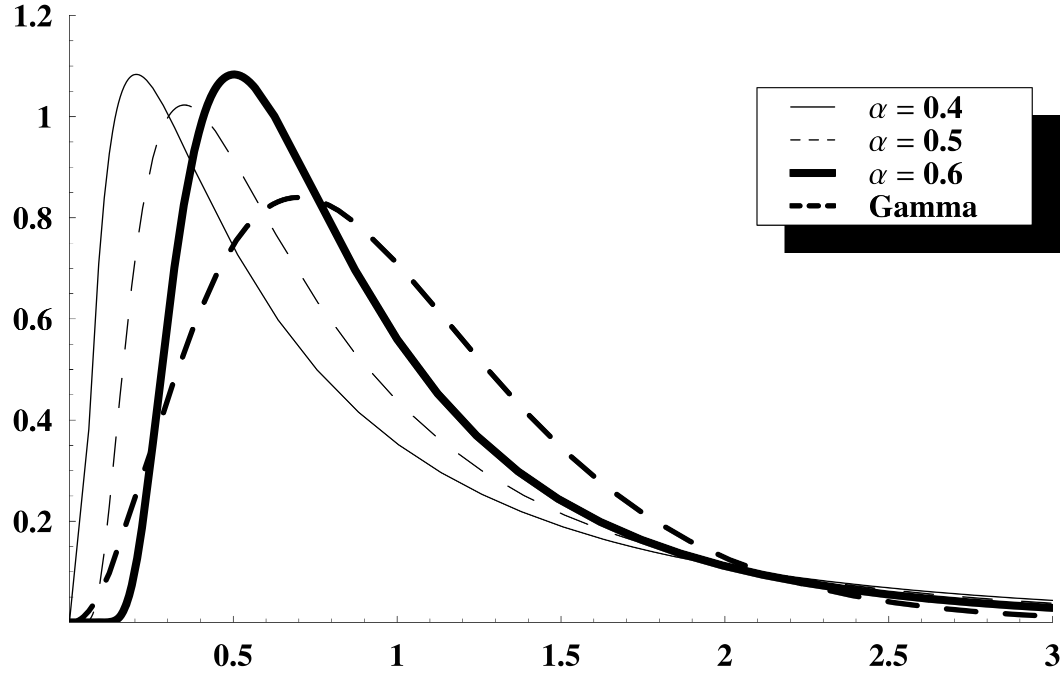
$$F_{\alpha,\sigma,\tau}(y) := \mathbb{E}[e^{-\tau Y} 1_{\{Y \leq y\}}] / \mathbb{E}[e^{-\tau Y}], \quad y \in \mathbb{R}.$$

Let $\Lambda \sim F_{\alpha,\sigma,\tau}$. Then for $\tau > 0$

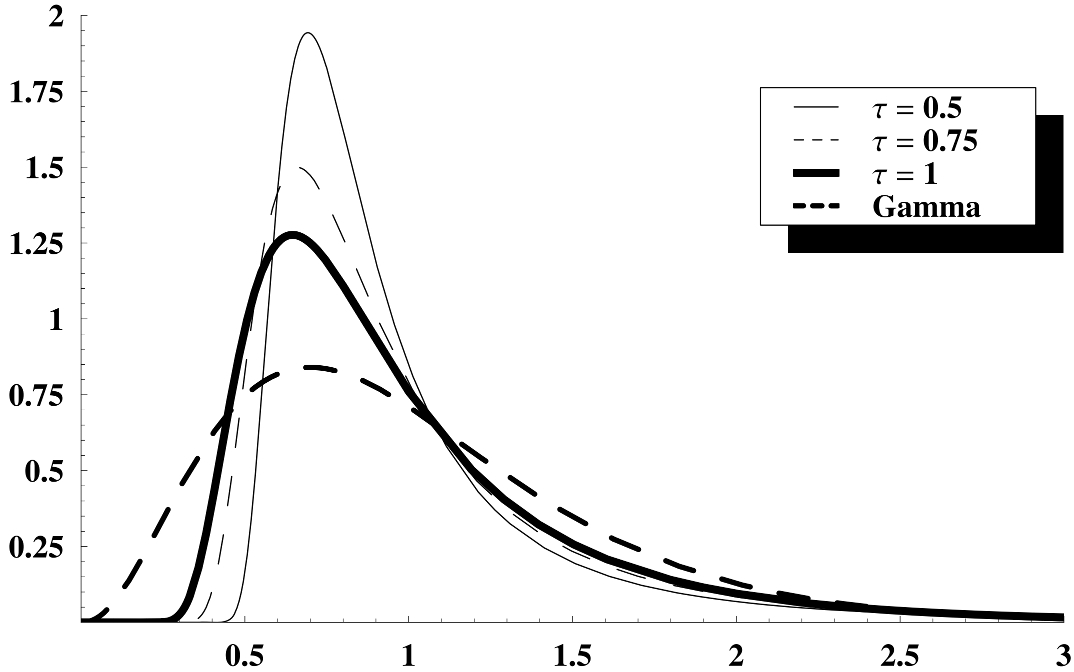
$$\mathcal{L}_{\Lambda}(s) = \exp\big(-\gamma_{\alpha,\sigma}\big((s + \tau)^{\alpha} - \tau^{\alpha}\big)\big), \quad s \geq -\tau,$$

$$\mathbb{E}[\Lambda] = -\mathcal{L}'_{\Lambda}(0) = \alpha\gamma_{\alpha,\sigma}\tau^{\alpha-1},$$

$$\text{Var}(\Lambda) = -\mathcal{L}''_{\Lambda}(0) - (\mathcal{L}'_{\Lambda}(0))^2 = \alpha(1 - \alpha)\gamma_{\alpha,\sigma}\tau^{\alpha-2}.$$



Density of $\Lambda \sim F_{\alpha,\sigma,\tau}$ in comparison with gamma distribution, where $\alpha \in (0,1)$ and $\sigma,\tau > 0$ satisfy $\mathbb{E}[\Lambda] = 1$ and $\mathrm{Var}(\Lambda) = 0.3$.



Density of $\Lambda \sim F_{\alpha,\sigma,\tau}$ in comparison with gamma distribution, where $\alpha \in (0,1)$ and $\sigma,\tau > 0$ satisfy $\mathbb{E}[\Lambda] = 1$ and $\mathrm{Var}(\Lambda) = 0.3$.

Application: Poisson–Tempered α-Stable Mixtures

For $\alpha \in (0,1)$, $\lambda, \sigma > 0$ and $\tau \geq 0$ consider $\Lambda \sim F_{\alpha,\sigma,\tau}$ and the Poisson mixture $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$.

Representation as compound Poisson distribution:

If

$$M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha))$$

and the i. i. d. sequence $\{N_m\}_{m\in\mathbb{N}}$ with

$$N_m \sim \text{ExtNegBin}\Big(-\alpha, 1, \frac{\tau}{\lambda+\tau}\Big)$$

are independent, then $N \stackrel{\text{d}}{=} N_1 + \cdots + N_M$.

Numerically Stable Algorithm for Poisson–Tempered α-Stable Mixtures

- Apply Panjer's recursion for $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$

$$\tilde{N} \sim \text{NegBin}(1-\alpha, p) \qquad \text{with} \qquad p = \frac{\tau}{\lambda+\tau}.$$

- Use weighted convolution to pass from

$$\tilde{N} \sim \text{NegBin}(1-\alpha, p) \ \rightarrow \ N \sim \text{ExtNegBin}(-\alpha, 1, \text{p}).$$

- Take the previous calculated distribution of $S = X_1 + \cdots + X_N$ as new claim size distribution and apply Panjer's recursion for

$$M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha)).$$

Example: Poisson–Lévy Mixture

Special case for $\alpha = 1/2$ and $\tau = 0$.

Lévy distribution: Given $\sigma > 0$, assume that

$$\Lambda = Y \sim f_{\mathsf{L}}(x) = \Big(\frac{\sigma}{2\pi x^3}\Big)^{1/2} \exp\Big(-\frac{\sigma}{2x}\Big), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.
Then $N \overset{\mathbf{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\lambda\sigma/2}\big)$$

and

$$N_m \sim \text{ExtNegBin}(-1/2, 1, 0), \quad m \in \mathbb{N},$$

and our numerically stable recursion is again applicable.

Example: Poisson–Inverse Gaussian Mixture

Fix $\mu, \tilde{\sigma} > 0$, define $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$.

Inverse Gaussian distribution: $\Lambda \sim F_{1/2,\sigma,\tau}$ has density

$$f_{\mathsf{IG}}(x) = \frac{\mu}{\sqrt{2\pi\tilde{\sigma}^2 x^3}} \exp\bigg(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2 x}\bigg), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.
Then $N \overset{\mathbf{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\sigma/2}(\sqrt{\lambda+\tau} - \sqrt{\tau})\big)$$

and

$$N_m \sim \text{ExtNegBin}\Big(-1/2, 1, \frac{\tau}{\lambda+\tau}\Big), \quad m \in \mathbb{N}.$$

Poisson–Reciprocal Inverse Gaussian Mixture

Fix $\mu, \tilde{\sigma} > 0$, define $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$.

Reciprocal inverse Gaussian distribution: Assume

$$\Lambda \sim f_{\mathrm{RIG}}(x) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 x}} \exp\Big(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2 x}\Big), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \mathrm{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.

Then $N \stackrel{\text{d}}{=} N_0 + N_1 + \cdots + N_M$ with independent

$$N_0 \sim \mathrm{NegBin}(1/2, p), \quad p = \tau/(\lambda + \tau),$$
$$M \sim \mathrm{Poisson}\big(\sqrt{\sigma/2}(\sqrt{\lambda+\tau} - \sqrt{\tau})\big),$$
$$N_m \sim \mathrm{ExtNegBin}(-1/2, 1, p), \quad m \in \mathbb{N}.$$

Further Applications of the Ideas

- Generalization of De Pril's recursion to calculate higher moments of $S = X_1 + \cdots + X_N$.
- Generalization of Panjer's recursion to claim sizes with mixed support (density and an atom at zero) $\rightarrow$ Integral equations for the density of S.

Reference

S. Gerhold, U. Schmock, R. Warnung:
A Generalization of Panjer's Recursion and Numerically Stable Risk Aggregation,
Accepted by Finance & Stochastics, available at
`http://www.fam.tuwien.ac.at/~schmock/Stable_Panjer_Recursion.html`