Efficient and Numerically Stable Aggregation of Dependent Risks

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Outline of Presentation

• Motivation
• CreditRisk+ and extensions
• Quantiles, expected shortfall, contributions to expected shortfall
• Application to operational risk
Software implementation:
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Extended version of talk available as lecture notes at:
http://www.fam.tuwien.ac.at/~schmock/notes/ExtensionsCreditRiskPlus.pdf

Motivation: Bernoulli Model for Defaults

• Bernoulli loss indicators
  \[ N_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults (within one year),} \\ 0 & \text{otherwise.} \end{cases} \]
• Default probability \( p_i = \mathbb{P}(N_i = 1) \) for \( i = 1, \ldots, m \).
• Random number of defaults \( N = N_1 + \cdots + N_m \).
• Probability distribution for \( n \in \{0, \ldots, m\} \)
  \[ \mathbb{P}(N = n) = \sum_{I} \mathbb{P}(N_i = 1 \text{ if } i = 1, \ldots, m) = \prod_{i \in I} p_i \prod_{i \in (1, \ldots, m) \setminus I} (1-p_i) \]
  \[ m = 1000, \quad n = 100 \quad \Rightarrow \quad \binom{1000}{100} \approx 6.4 \times 10^{139} \text{ terms} \]

Observations...

• Already the Bernoulli model with independent loss indicators has far too many terms for the calculation of the portfolio loss distribution in the general case.
• In the general Bernoulli mixture model, individual terms are too complicated to compute numerically.
• Different exposures and recovery rates are not even considered.

... and Conclusions

• Simplifying assumptions are necessary.
• Approximations need to be considered.
Poisson Approximation

- $X_1, \ldots, X_m$ independent default 0-1-indicators
- Intensity $\lambda = \sum_{i=1}^{m} p_i$ with $p_i = \mathbb{P}(X_i = 1)$
- Number of default events $W = \sum_{i=1}^{m} X_i$
- Total variation distance
  \[ d_{TV}(\mu, \nu) = \sup_{A \subset \mathbb{N}_0} |\mu(A) - \nu(A)| \]

Quality of Poisson approximation (Barbour/Hall, 1984):

\[ d_{TV}(L(W), \text{Poisson}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{m} p_i^2 \]

For full proof with Stein–Chen method, see e.g. Barbour, Holst and Janson: Poisson Approximation, Clarendon Press (1992).

Simple Poisson Model for Defaults

- Number $N_i$ of defaults of obligor $i \in \{1, \ldots, m\}$
- Assume $N_i \sim \text{Poisson}(\lambda_i)$ for all $i \in \{1, \ldots, m\}$ (several defaults of an obligor possible).
- Assume independence of $N_1, \ldots, N_m$.
- Random number of defaults $N = N_1 + \cdots + N_m$.
- $N \sim \text{Poisson}(\lambda)$ with $\lambda = \lambda_1 + \cdots + \lambda_m$, i.e.,
  \[ \mathbb{P}(N = n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{for all } n \in \mathbb{N}_0. \]
- $m = 20$, $\lambda_i = 0.2$ \implies $\mathbb{P}(N > 20) \leq 2 \times 10^{-9}$.

Introduction to CreditRisk+, Standard Features

- Developed by Credit Suisse First Boston.
- Actuarial model for the aggregation of credit risks.
- Based on the Poisson approximation of individual defaults and the divisibility of the Poisson distribution.
- Allows for deterministic exposures/recovery rates.
- Several independent risk factors for dependence of default frequencies can be considered.
- Probability generating function $\varphi_L$ of the credit portfolio loss $L$ is available in closed form.
  \[ \rightarrow \text{No Monte Carlo simulation, no stochastic error!} \]

Extensions of CreditRisk+

- Stochastic losses of individual obligors are allowed, distribution may depend on the causing risk factor.
- Risk groups with dependent stochastic losses given default are possible.
- Risk factors for default frequencies may be dependent.
- Risk contributions of obligors can be calculated.
- Even with all the extensions, the probability generating function $\varphi_L$ of the credit portfolio loss $L$ is available in closed form.
  \[ \rightarrow \text{No Monte Carlo simulation, no stochastic error!} \]
- Distribution of $L$ and risk contributions can be calculated from $\varphi_{\gamma,L}$ with a numerically stable algorithm.
Input Parameters of CreditRisk$^+$ (Extended Version)

- Number of obligors $m \in \mathbb{N}$.
- Basic loss unit $E > 0$.
- Number $K \in \mathbb{N}_0$ of risk factors or non-idiocentric, (independent) default causes.
- Relative default variances $\sigma_k^2 > 0$ of risk factors $k \in \{1, \ldots, K\}$.
- Collection $G$ of nonempty subsets of all obligors $\{1, \ldots, m\}$, called risk groups.

Further Assumptions, Notation

- We assume that every obligor $i \in \{1, \ldots, m\}$ belongs to at least one group $g \in G$.
- Let $G_i := \{g \in G \mid i \in g\}$ denote the set of all risk groups to which obligor $i \in \{1, \ldots, m\}$ belongs, by assumption $G_i \neq \emptyset$.
- We assume that for each group the susceptibilities (also called weights) exhaustively describe the risk factors. That is, for all $g \in G$,
  \[ \sum_{k=0}^{K} w_{g,k} = 1. \]

Input Parameters of CreditRisk$^+$ (Cont.)

For every group $g \in G$ we need

- the (one year) default probability $p_g \in [0, 1]$,
- the susceptibility $w_{g,0} \in [0, 1]$ to idiosyncratic risk,
- the susceptibilities $w_{g,k} \in [0, 1]$ to risk factors $k \in \{1, \ldots, K\}$,
- the multivariate probability distributions $Q_{g,k} = \{q_{g,k,\mu} \mid \mu \in \mathbb{N}_0^g\}$ on $\mathbb{N}_0^g$ describing the stochastic losses of all the obligors $i \in g$ in multiples of the basic loss unit $E$ in case the risk group $g$ defaults due to risk $k \in \{0, \ldots, K\}$.

Notation for Default Events of Risk Groups

Number of defaults for every risk group $g \in G$:  
- $N_{g,0}$ due to idiosyncratic risk,
- $N_{g,k}$ due to risk $k \in \{1, \ldots, K\}$,
- $N_g := \sum_{k=0}^{K} N_{g,k}$ total.

Notation for Default Events of Individual Obligors

Number of defaults for every obligor $i \in \{1, \ldots, m\}$
- $N_{i,0} := \sum_{g \in G_i} N_{g,0}$ due to idiosyncratic risk,
- $N_{i,k} := \sum_{g \in G_i} N_{g,k}$ due to risk $k \in \{1, \ldots, K\}$,
- $N_i := \sum_{k=0}^{K} N_{i,k} = \sum_{g \in G_i} N_g$ total.
**Notation for Stochastic Losses**

Loss at default number $n \in \mathbb{N}$ of risk group $g \in G$ due to risk factor $k \in \{1, \ldots, K\}$ or idiosyncratic risk $k = 0$

- $L_{g,i,k,n}$ part attributed to obligor $i \in g$
- $L_{g,k,n} := \sum_{i \in g} L_{g,i,k,n}$ loss of entire group

Summation over default numbers, risks and groups:

- $L_{g,k} := \sum_{n=1}^{N_{g,k}} L_{g,k,n}$ total loss of the group for risk $k$
- $L_{g} := \sum_{k=0}^{K} L_{g,k}$ total of the risk group
- $L := \sum_{g \in G} L_{g}$ portfolio loss

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**Probabilistic Assumptions**

**for the Extended Version of CreditRisk+$^+$**

- For every group $g \in G$ and every risk $k \in \{0, \ldots, K\}$, the sequence of $\mathbb{N}_0^g$-valued random vectors $(L_{g,i,k,n})_{i \in g}$ with $n \in \mathbb{N}$ is i.i.d. and independent of all other random variables, with distribution

  \[ \mathbb{P}(L_{g,i,k,1} = \mu_i \text{ for all } i \in g) = q_{g,k,\mu}, \quad \mu \in \mathbb{N}_0^g. \]

- For each group $g \in G$, the number $N_{g,0}$ of idiosyncratic defaults is Poisson distributed according to the Poisson intensity $\lambda_g$ and the susceptibility $w_{g,0}$, i.e.,

  \[ N_{g,0} \sim \text{Poission}(\lambda_g w_{g,0}) \quad \text{for every } g \in G. \]

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**Probabilistic Assumptions (Cont.)**

- The group default numbers $\{N_{g,k}\}_{g \in G}$ due to idiosyncratic risk are independent from one another and from all other random variables.
- The risks factors $\Lambda_1, \ldots, \Lambda_K$ are independent, each one gamma distributed with $\mathbb{E}[\Lambda_k] = 1$ and $\text{Var}(\Lambda_k) = \sigma_k^2 > 0$, i.e., $\alpha_k = \beta_k = 1/\sigma_k^2$.
- For all groups $g \in G$ and risks $k \in \{1, \ldots, K\}$,

  \[ \mathcal{L}(N_{g,k} | \Lambda_1, \ldots, \Lambda_K) \overset{\text{a.s.}}{=} \mathcal{L}(N_{g,k} | \Lambda_k) \overset{\text{a.s.}}{=} \text{Poission}(\lambda_g w_{g,k} \Lambda_k). \]

- Conditionally on $\Lambda_1, \ldots, \Lambda_K$, the risk factor based defaults $\{N_{g,k} | g \in G, k \in \{1, \ldots, K\}\}$ are independent.
**Weighted Probability Generating Function**

In order to calculate terms needed for the risk contributions we will need what we call weighted probability generating functions.

**Definition:** For $L : \Omega \rightarrow \mathbb{N}_0$ and an integrable random variable $X : \Omega \rightarrow \mathbb{R}$, we define the $X$-weighted probability generating function by

$$\varphi_{L,X}(s) = \mathbb{E}[Xs^L] = \sum_{n=0}^{\infty} \mathbb{E}[X1\{L=n\}] s^n,$$

which is meaningful at least for all $s \in \mathbb{C}$ with $|s| \leq 1$.

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**Weighted Probability Generating Function (Cont.)**

We will need expressions of the form $\mathbb{E}[\Lambda_k 1\{L=n\}]$ for $k \in \{1, \ldots, K\}$ and $n \in \mathbb{N}_0$, which can be derived by

$$\varphi_{L,\Lambda_k}^{(n)}(0) = n! \mathbb{E}[\Lambda_k 1\{L=n\}] .$$

Unifying approach for the $\gamma$-weighted probability generating function of the loss:

Fix $\gamma = (\gamma_1, \ldots, \gamma_K) \in [0, \infty)^K$ and define

$$\varphi_{L,\gamma}(s) := \mathbb{E}[\Lambda_1^{\gamma_1} \cdots \Lambda_K^{\gamma_K} s^L], \quad |s| \leq 1,$$

for the risk factors $\Lambda_1, \ldots, \Lambda_K$ and the total loss $L$.

$\gamma = 0$ gives the probability generating function $\varphi_L$ of $L$.

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**The Closed Form of the WPGF**

$$\varphi_{L,\gamma}(s) = C_{\gamma} \exp\left(\bar{\lambda}_0(\varphi_0(s) - 1) - \sum_{k=1}^{K} \left(\frac{1}{\sigma_k^2} + \gamma_k\right) \log(1 - \bar{\lambda}_k \sigma_k^2(\varphi_k(s) - 1))\right),$$

where $C_{\gamma} := \prod_{k=1}^{K} \mathbb{E}[^{\gamma_k}L_k] = 1$ if all $\gamma_k \in \{0, 1\}$, with PGF of mixture distributions (conditioned to be positive)

$$\varphi_k(s) := \sum_{g \in G} \frac{\lambda_g w_{g,k}}{\lambda_k} \varphi_{L_g,1}(s), \quad \bar{\lambda}_k := \sum_{g \in G} \lambda_g w_{g,k}(1-q_{g,k,0}).$$


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**Distributions in the Panjer Class**

Definition: A probability distribution $\{q_n\}_{n \in \mathbb{N}_0}$ is said to belong to the Panjer($a, b, k$) class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \cdots = q_{k-1} = 0$ and

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1} \text{ for all } n \in \mathbb{N} \text{ with } n \geq k + 1.$$

Important Examples: (all distributions are known)

- Poisson($\lambda$) $\in$ Panjer($0, \lambda, 0$) with $\lambda > 0$
- NegBin($\alpha, p$) $\in$ Panjer($q, (\alpha - 1)q, 0$)
- Log($q$) $\in$ Panjer($q, -q, 1$) with $q \in (0, 1)$ and

$$q_n = -\frac{q^n}{n \log(1-q)} \text{ for all } n \in \mathbb{N}.$$
Extended Panjer Recursion

If $L(N) \in \text{Panjer}(a, b, k)$, independent of the i. i. d. $N_0$-valued sequence $\{X_n\}_{n \in \mathbb{N}}$, and $aP(X_1 = 0) \neq 1$, then $S := X_1 + \cdots + X_N$ satisfies

$$P(S = 0) = \varphi_N(P(X_1 = 0))$$

with $\varphi_N$ probability generating function of $N$, and

$$P(S = n) = \frac{1}{1 - aP(X_1 = 0)} \left( P(S_k = n)P(N = k) + \sum_{j=1}^{n} \left( a + \frac{bj}{n} \right) P(X_1 = j)P(S = n - j) \right)$$

for all $n \in \mathbb{N}$, where $S_k = X_1 + \cdots + X_k$.

Application of Extended Panjer Recursion

Remark: Recursion scheme is numerically stable for Poisson($\lambda$), NegBin($\alpha, p$), and Log($q$).

Observation: If $M \sim \text{Poisson}(\lambda)$, independent of the i. i. d. sequence $\{C_n\}_{n \in \mathbb{N}}$ with $C_1 \sim \text{Log}(q)$, then

$$N = C_1 + \cdots + C_M \sim \text{NegBin} \left( \frac{-\lambda \log(1 - q)}{\log(1 - q)}, 1 - q \right).$$

Application: Calculate

- Mixture distribution $\varphi_k$ for risks $k \in \{0, \ldots, K\}$.
- Panjer recursion for log. dist. for risks $k \in \{1, \ldots, K\}$.
- Mixture distribution of $\varphi_0$ and recursion results.
- Final Panjer recursion for Poisson distribution.
Measuring Risk by Quantiles

Let $X$ be a loss variable and $\delta \in (0, 1)$ a level.

**Definition:** Lower $\delta$-quantile of $X$

$$q_\delta(X) := \min\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \delta\}.$$

**Remark:** Quantiles are used as value-at-risk, they have bad properties concerning diversification.

**Properties:** $q_\delta(X)$ can jump when
- the level $\delta$ varies slightly,
- the loss variable $X$ varies slightly.

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Calculating Expected Shortfall in CreditRisk+

- Credit portfolio loss $L$ is a discrete random variable, → More complicated definition has to be used.
- The lower quantile $q_\delta(L)$ and $\mathbb{P}(L \leq q_\delta(L))$ can be calculated using the CreditRisk+ algorithm.
- Furthermore $\mathbb{E}[L1_{\{L > q_\delta(L)\}}] = \mathbb{E}[L] - \mathbb{E}[L1_{\{L \leq q_\delta(L)\}}]$ with
  $$\mathbb{E}[L] = \sum_{g \in G} \sum_{k=0}^{K} \lambda_g w_{g,k} \mathbb{E}[L_{g,k,1}]$$
  and
  $$\mathbb{E}[L1_{\{L \leq q_\delta(L)\}}] = \sum_{l=1}^{q_\delta(L)} l \mathbb{P}(L = l).$$
Contributions to Expected Shortfall – Definition

**Definition:** For a subportfolio loss $X$ with $X \in L^1(\mathbb{P})$ within a portfolio loss $Y$ define the expected shortfall contribution at level $\delta \in (0, 1)$ of $X$ to $Y$ by

$$ES_\delta[X, Y] = \frac{E[X1_{\{Y > q_\delta(Y)\}}] + \beta_Y E[X1_{\{Y = q_\delta(Y)\}}]}{1 - \delta}$$

where

$$\beta_Y = \frac{P(Y \leq q_\delta(Y)) - \delta}{P(Y = q_\delta(Y))}$$

if $P(Y = q_\delta(Y)) > 0$ and 0 otherwise.

**Remark:** If $P(Y \leq q_\delta(Y)) = \delta$, then $\beta_Y = 0$ and $ES_\delta[X, Y] = E[X | Y > q_\delta(Y)]$.

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**Lemma on Risk Contributions in CreditRisk**

For every obligor $i \in \{1, \ldots, m\}$, every group $g \in G_i$ and total loss $l \in \mathbb{N}_0$,

$$E[L_{g, i, 0}1_{\{L=l\}}] = \lambda_g w_{g, 0} \sum_{\nu=1}^l E[L_{g, i, 0}1_{\{L=g, 0, i=\nu\}}] P(L = l - \nu)$$

and, for every risk $k \in \{1, \ldots, K\}$,

$$E[L_{g, i, k}1_{\{L=l\}}] = \lambda_g w_{g, k} \sum_{\nu=1}^l E[L_{g, i, k}1_{\{L=g, k, i=\nu\}}] E[\Lambda_k1_{\{L=l-\nu\}}].$$

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**Possible Approaches to Operational Risk Modelling**

The Basel committee defined three approaches towards the quantification of operational risk. The two simple ones define concrete formulae for the risk capital, namely

- Basic indicator approach (BIA).
- Standardized approach (SA).

To reduce supervisory capital needs, an individual

- Advanced measurement approach (AMA) can be chosen.

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Business Lines for Operational Risk

- Eight business lines in the standardized approach:
  1. Corporate finance,
  2. Trading & sales,
  3. Retail banking,
  4. Commercial banking,
  5. Payment & settlement,
  6. Agency services,
  7. Asset management,
  8. Retail brokerage.

- These business lines also serve as categories for an advanced measurement approach.

Seven Loss-Types Distinguished for the Advanced Measurement Approach

- Internal fraud
- External fraud
- Employment practices & workplace safety
- Clients, products & business practice
- Damage to physical assets
- Business disruption & system failures
- Execution, delivery & process management

Application of Extended CreditRisk+ Methodology to Operational Risk: Reinterpretation of the Credit Risk Notation

- Number $m$ of obligors $\mapsto$ number of business lines ($m = 8$ for the ones given in the Basel committee’s document is an appropriate choice).
- Basic loss unit $E$ stays the same ($E = 10000$).
- Number $K$ of non-ideosyncratic risk factors $\mapsto$ number of loss types ($K = 7$ for above list).
- Numbers $\sigma^2_k > 0$ denote the relative variance of occurrences of losses of type $k \in \{1, \ldots, K\}$.
- $G$ contains the subsets of all business lines which can incur a loss due to the same event.

Notation for Business Lines and Risk Groups

We need for every risk group $g \in G$ of business lines
- the (one year) intensity $\lambda_g \geq 0$ for being hit by an operational loss event,
- the conditional probability $w_{g,0} \in [0,1]$ for an idiosyncratic operational loss event not to belong to the types in $\{1, \ldots, K\}$, of course $w_{g,0} = 0$ is a possible choice,
- the conditional probabilities $w_{g,k} \in [0,1]$ for an operational loss event to be of type $k \in \{1, \ldots, K\}$,
the multivariate probability distribution $Q_{g,k} = \{q_{g,k,\mu}\}_{\mu \in \mathbb{N}_0^g}$ on $\mathbb{N}_0^g$ describing the severity of the stochastic losses of the business lines $i \in g$ in case an operational loss event of type $k \in \{0, \ldots, K\}$ hits the group $g$ of business lines.

Operational Risk Management

With the adoption of the extended CreditRisk$^+$ model for operational risk, a risk manager can

- calculate the distribution of the operational loss,
- calculate risk measures such as value-at-risk and expected shortfall (might be infinity)
- and identify risky business lines and groups by their risk contribution (in case of finite expected shortfall).

Further Extensions of CreditRisk$^+$

- Dependent risk factors, possibly with an interactive and hierarchical structure
- Other mixture distributions besides the gamma distribution
- Special choices of weighted expected shortfall as risk measure and corresponding risk contribution