

# Modelling and Aggregation of Dependent Credit and Operational Risks

Risk Day 2006  
ETH Zürich, Oct. 20, 2006

Prof. Dr. Uwe Schmock  
Dipl.-Ing. Richard Warnung  
Financial and Actuarial Mathematics  
Vienna University of Technology, Austria  
[www.fam.tuwien.ac.at](http://www.fam.tuwien.ac.at)

## Outline of Presentation

- Motivation
- CreditRisk<sup>+</sup> and extensions
- Quantiles, expected shortfall, contributions to expected shortfall
- Application to operational risk

Software implementation:

- Dipl.-Ing. Severin Resch
- Dipl.-Ing. Richard Warnung

Extended version of talk available as lecture notes at:  
<http://www.fam.tuwien.ac.at/~schmock/notes/ExtentionsCreditRiskPlus.pdf>

©Oct. 20, 2006, U. Schmock, FAM, TU Vienna

2

## Motivation: Bernoulli Model for Defaults

- Bernoulli loss indicators

$$N_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults (within one year),} \\ 0 & \text{otherwise.} \end{cases}$$

- Default probability  $p_i = \mathbb{P}(N_i = 1)$  for  $i = 1, \dots, m$ .
- Random number of defaults  $N = N_1 + \dots + N_m$ .
- Probability distribution for  $n \in \{0, \dots, m\}$

$$\mathbb{P}(N = n) = \sum_{\substack{I \subset \{1, \dots, m\} \\ |I|=n}} \underbrace{\mathbb{P}(N_i = 1_{I(i)} \text{ for } i = 1, \dots, m)}_{\text{if ind. } (\prod_{i \in I} p_i) \prod_{i \in \{1, \dots, m\} \setminus I} (1-p_i)}$$

$$m = 1000, n = 100 \implies \binom{1000}{100} \approx 6.4 \times 10^{139} \text{ terms}$$

©Oct. 20, 2006, U. Schmock, FAM, TU Vienna

3

## Observations ...

- Already the Bernoulli model with independent loss indicators has far too many terms for the calculation of the portfolio loss distribution in the general case.
- In the general Bernoulli mixture model, individual terms are too complicated to compute numerically.
- Different exposures and recovery rates are not even considered.

## ... and Conclusions

- Simplifying assumptions are necessary.
- Approximations need to be considered.

©Oct. 20, 2006, U. Schmock, FAM, TU Vienna

4

## Poisson Approximation

- $X_1, \dots, X_m$  independent default 0-1-indicators
- Intensity  $\lambda = \sum_{i=1}^m p_i$  with  $p_i = \mathbb{P}(X_i = 1)$
- Number of default events  $W = \sum_{i=1}^m X_i$
- Total variation distance

$$d_{TV}(\mu, \nu) = \sup_{A \subset \mathbb{N}_0} |\mu(A) - \nu(A)|$$

Quality of Poisson approximation (Barbour/Hall, 1984):

$$d_{TV}(\mathcal{L}(W), \text{Poisson}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^m p_i^2$$

For full proof with Stein–Chen method, see e.g. Barbour, Holst and Janson: *Poisson Approximation*, Clarendon Press (1992).

## Introduction to CreditRisk<sup>+</sup>, Standard Features

- Developed by Credit Suisse First Boston.
- Actuarial model for the aggregation of credit risks.
- Based on the Poisson approximation of individual defaults and the divisibility of the Poisson distribution.
- Allows for deterministic exposures/recovery rates.
- Several independent risk factors for dependence of default frequencies can be considered.
- Probability generating function  $\varphi_L$  of the credit portfolio loss  $L$  is available in closed form.  
→ No Monte Carlo simulation, no stochastic error!

## Simple Poisson Model for Defaults

- Number  $N_i$  of defaults of obligor  $i \in \{1, \dots, m\}$
- Assume  $N_i \sim \text{Poisson}(\lambda_i)$  for all  $i \in \{1, \dots, m\}$  (several defaults of an obligor possible).
- Assume independence of  $N_1, \dots, N_m$ .
- Random number of defaults  $N = N_1 + \dots + N_m$ .
- $N \sim \text{Poisson}(\lambda)$  with  $\lambda = \lambda_1 + \dots + \lambda_m$ , i.e.,

$$\mathbb{P}(N = n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{for all } n \in \mathbb{N}_0.$$

- $m = 20, \lambda_i = 0.2 \implies \mathbb{P}(N > 20) \leq 2 \times 10^{-9}$ .

## Extensions of CreditRisk<sup>+</sup>

- Stochastic losses of individual obligors are allowed, distribution may depend on the causing risk factor.
- Risk groups with dependent stochastic losses given default are possible.
- Risk factors for default frequencies may be dependent.
- Risk contributions of obligors can be calculated.
- Even with all the extensions, the probability generating function  $\varphi_L$  of the credit portfolio loss  $L$  is available in closed form.  
→ No Monte Carlo simulation, no stochastic error!
- Distribution of  $L$  and risk contributions can be calculated from  $\varphi_{\gamma, L}$  with a numerically stable algorithm.

## Input Parameters of CreditRisk<sup>+</sup> (Extended Version)

- Number of obligors  $m \in \mathbb{N}$ .
- Basic loss unit  $E > 0$ .
- Number  $K \in \mathbb{N}_0$  of risk factors or non-idiosyncratic, (independent) default causes.
- Relative default variances  $\sigma_k^2 > 0$  of risk factors  $k \in \{1, \dots, K\}$ .
- Collection  $G$  of nonempty subsets of all obligors  $\{1, \dots, m\}$ , called risk groups.

## Further Assumptions, Notation

- We assume that every obligor  $i \in \{1, \dots, m\}$  belongs to at least one group  $g \in G$ .
- Let  $G_i := \{g \in G \mid i \in g\}$  denote the set of all risk groups to which obligor  $i \in \{1, \dots, m\}$  belongs, by assumption  $G_i \neq \emptyset$ .
- We assume that for each group the susceptibilities (also called weights) exhaustively describe the risk factors. That is, for all  $g \in G$ ,

$$\sum_{k=0}^K w_{g,k} = 1.$$

## Input Parameters of CreditRisk<sup>+</sup> (Cont.)

For every group  $g \in G$  we need

- the (one year) default probability  $p_g \in [0, 1]$ ,
- the susceptibility  $w_{g,0} \in [0, 1]$  to idiosyncratic risk,
- the susceptibilities  $w_{g,k} \in [0, 1]$  to risk factors  $k \in \{1, \dots, K\}$ ,
- the multivariate probability distributions  $Q_{g,k} = \{q_{g,k,\mu}\}_{\mu \in \mathbb{N}_0^g}$  on  $\mathbb{N}_0^g$  describing the stochastic losses of all the obligors  $i \in g$  in multiples of the basic loss unit  $E$  in case the risk group  $g$  defaults due to risk  $k \in \{0, \dots, K\}$ .

## Notation for Default Events of Risk Groups

Number of defaults for every risk group  $g \in G$ :

- $N_{g,0}$  due to idiosyncratic risk,
- $N_{g,k}$  due to risk  $k \in \{1, \dots, K\}$ ,
- $N_g := \sum_{k=0}^K N_{g,k}$  total.

## Notation for Default Events of Individual Obligators

Number of defaults for every obligor  $i \in \{1, \dots, m\}$

- $N_{i,0} := \sum_{g \in G_i} N_{g,0}$  due to idiosyncratic risk,
- $N_{i,k} := \sum_{g \in G_i} N_{g,k}$  due to risk  $k \in \{1, \dots, K\}$ ,
- $N_i := \sum_{k=0}^K N_{i,k} = \sum_{g \in G_i} N_g$  total.

## Notation for Stochastic Losses

Loss at default number  $n \in \mathbb{N}$  of risk group  $g \in G$  due to risk factor  $k \in \{1, \dots, K\}$  or idiosyncratic risk  $k = 0$

- $L_{g,i,k,n}$  part attributed to obligor  $i \in g$
- $L_{g,k,n} := \sum_{i \in g} L_{g,i,k,n}$  loss of entire group

Summation over default numbers, risks and groups:

- $L_{g,k} := \sum_{n=1}^{N_{g,k}} L_{g,k,n}$  total loss of the group for risk  $k$
- $L_g := \sum_{k=0}^K L_{g,k}$  total of the risk group
- $L = \sum_{g \in G} L_g$  portfolio loss

## Probabilistic Assumptions for the Extended Version of CreditRisk<sup>+</sup>

- For every group  $g \in G$  and every risk  $k \in \{0, \dots, K\}$ , the sequence of  $\mathbb{N}_0^g$ -valued random vectors  $(L_{g,i,k,n})_{i \in g}$  with  $n \in \mathbb{N}$  is i.i.d. and independent of all other random variables, with distribution

$$\mathbb{P}(L_{g,i,k,1} = \mu_i \text{ for all } i \in g) = q_{g,k,\mu}, \quad \mu \in \mathbb{N}_0^g.$$

- For each group  $g \in G$ , the number  $N_{g,0}$  of idiosyncratic defaults is Poisson distributed according to the Poisson intensity  $\lambda_g$  and the susceptibility  $w_{g,0}$ , i.e.,

$$N_{g,0} \sim \text{Poisson}(\lambda_g w_{g,0}) \quad \text{for every } g \in G.$$

## Loss Attributed to Obligor $i \in \{1, \dots, m\}$

- Due to group  $g \in G_i$  and risk  $k \in \{0, \dots, K\}$

$$L_{g,i,k} := \sum_{n=1}^{N_{g,k}} L_{g,i,k,n}.$$

- Due to risk  $k \in \{0, \dots, K\}$

$$L_{i,k} := \sum_{g \in G_i} L_{g,i,k}.$$

- Total attributed loss

$$L_i := \sum_{k=0}^K L_{i,k}.$$

## Probabilistic Assumptions (Cont.)

- The group default numbers  $\{N_{g,0}\}_{g \in G}$  due to idiosyncratic risk are independent from one another and from all other random variables.
- The risks factors  $\Lambda_1, \dots, \Lambda_K$  are independent, each one gamma distributed with  $\mathbb{E}[\Lambda_k] = 1$  and  $\text{Var}(\Lambda_k) = \sigma_k^2 > 0$ , i.e.,  $\alpha_k = \beta_k = 1/\sigma_k^2$ .
- For all groups  $g \in G$  and risks  $k \in \{1, \dots, K\}$ ,

$$\begin{aligned} \mathcal{L}(N_{g,k} | \Lambda_1, \dots, \Lambda_K) &\stackrel{\text{a.s.}}{=} \mathcal{L}(N_{g,k} | \Lambda_k) \\ &\stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda_g w_{g,k} \Lambda_k). \end{aligned}$$

- Conditionally on  $\Lambda_1, \dots, \Lambda_K$ , the risk factor based defaults  $\{N_{g,k} | g \in G, k \in \{1, \dots, K\}\}$  are independent.

## Weighted Probability Generating Function

In order to calculate terms needed for the risk contributions we will need what we call weighted probability generating functions.

Definition: For  $L : \Omega \rightarrow \mathbb{N}_0$  and an integrable random variable  $X : \Omega \rightarrow \mathbb{R}$ , we define the  $X$ -weighted probability generating function by

$$\varphi_{L,X}(s) = \mathbb{E}[Xs^L] = \sum_{n=0}^{\infty} \mathbb{E}[X1_{\{L=n\}}] s^n,$$

which is meaningful at least for all  $s \in \mathbb{C}$  with  $|s| \leq 1$ .

## The Closed Form of the WPGF

Using the probabilistic assumptions and properties of the Poisson and the Gamma distribution we arrive at

$$\varphi_{L,\gamma}(s) = C_\gamma \exp\left(\psi_0(s) - \sum_{k=1}^K \left(\frac{1}{\sigma_k^2} + \gamma_k\right) \log(1 - \sigma_k^2 \psi_k(s))\right),$$

where  $C_\gamma := \prod_{k=1}^K \mathbb{E}[\Lambda_k^{\gamma_k}] = 1$  if all  $\gamma_k \in \{0, 1\}$ , which are the cases which we will consider later on and

$$\psi_k(s) := \sum_{g \in G} \lambda_g w_{g,k} (\varphi_{L_{g,k,1}}(s) - 1), \quad k \in \{0, \dots, K\}.$$

Numerical inversion similar to: H. Haaf, O. Reiß, J. Schoenmakers, *Numerically Stable Computation of CreditRisk<sup>+</sup>*, 2003.

## Weighted Probability Generating Function (Cont.)

We will need expressions of the form  $\mathbb{E}[\Lambda_k 1_{\{L=n\}}]$  for  $k \in \{1, \dots, K\}$  and  $n \in \mathbb{N}_0$ , which can be derived by

$$\varphi_{L,\Lambda_k}^{(n)}(0) = n! \mathbb{E}[\Lambda_k 1_{\{L=n\}}].$$

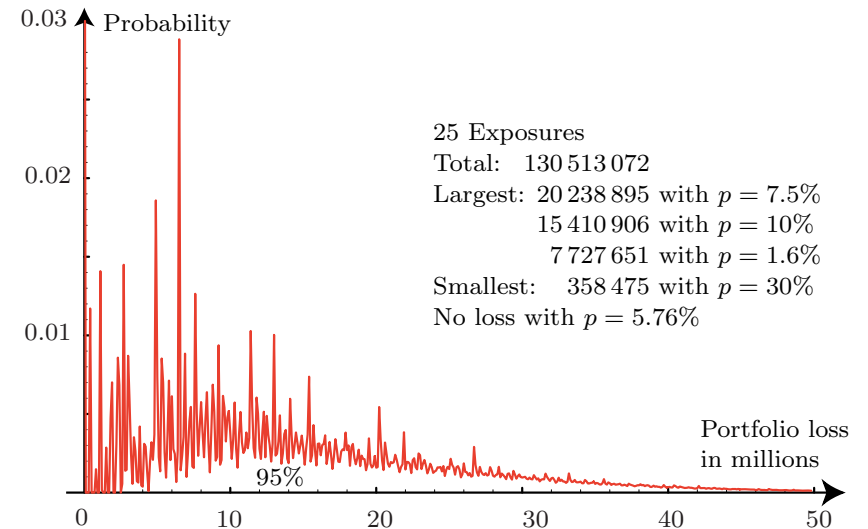
Unifying approach for the  $\gamma$ -weighted probability generating function of the loss:

Fix  $\gamma = (\gamma_1, \dots, \gamma_K) \in [0, \infty)^K$  and define

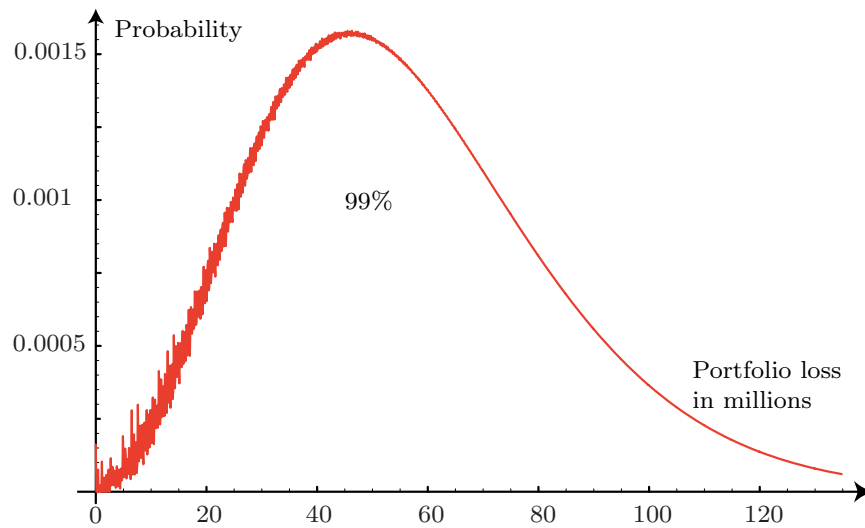
$$\varphi_{L,\gamma}(s) := \mathbb{E}[\Lambda_1^{\gamma_1} \dots \Lambda_K^{\gamma_K} s^L], \quad |s| \leq 1,$$

for the risk factors  $\Lambda_1, \dots, \Lambda_K$  and the total loss  $L$ .

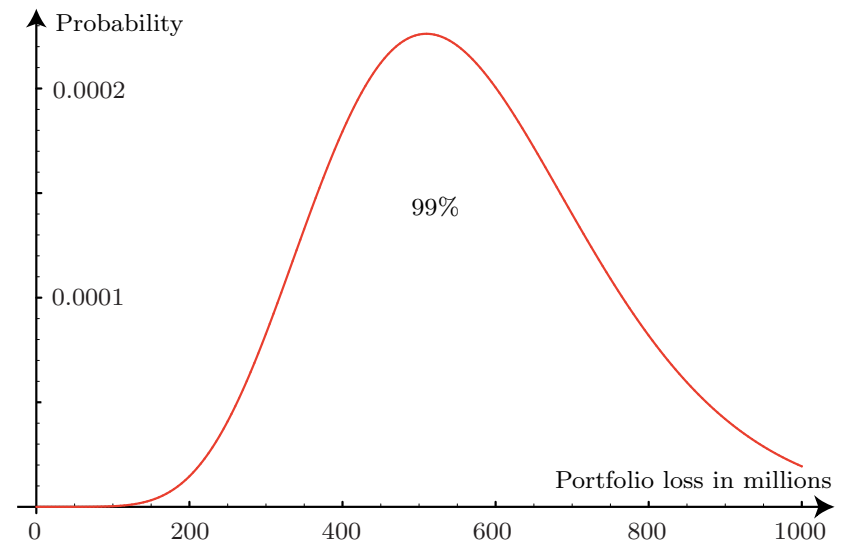
$\gamma = 0$  gives the probability generating function  $\varphi_L$  of  $L$ .



Loss distribution in a credit portfolio of 25 exposures calculated with CreditRisk<sup>+</sup> and basic loss unit 100 000.



Loss distribution in a credit portfolio of 100 exposures calculated with CreditRisk<sup>+</sup> and basic loss unit 100 000.



Loss distribution in a credit portfolio of 1000 exposures calculated with CreditRisk<sup>+</sup> and basic loss unit 100 000.

## Measuring Risk by Quantiles

Let  $X$  be a loss variable and  $\delta \in (0, 1)$  a level.

**Definition:** Lower  $\delta$ -quantile of  $X$

$$q_\delta(X) := \min\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \delta\}.$$

**Remark:** Quantiles are used as value-at-risk, they have bad properties concerning diversification.

**Properties:**  $q_\delta(X)$  can jump when

- the level  $\delta$  varies slightly,
- the loss variable  $X$  varies slightly.

## Measuring Risk by Expected Shortfall

Let  $X$  be a loss variable and  $\delta \in (0, 1)$  a level.

**Definition:** The expected shortfall is defined as

$$ES_\delta[X] := \frac{\mathbb{E}[X 1_{\{X > q_\delta(X)\}}] + q_\delta(X)(\mathbb{P}(X \leq q_\delta(X)) - \delta)}{1 - \delta}.$$

**Remark:** If  $\mathbb{P}(X \leq q_\delta(X)) = \delta$ , in particular if the distribution function  $\mathbb{R} \ni x \mapsto \mathbb{P}(X \leq x)$  of  $X$  is also left-continuous at  $x = q_\delta(X)$ , then

$$ES_\delta[X] = E[X \mid X > q_\delta(X)].$$

## Calculation of Expected Shortfall in CreditRisk<sup>+</sup>

- Credit portfolio loss  $L$  is a discrete random variable,  
→ More complicated definition has to be used.
- The lower quantile  $q_\delta(L)$  and  $\mathbb{P}(L \leq q_\delta(L))$  can be calculated using the CreditRisk<sup>+</sup> algorithm.
- Furthermore  $\mathbb{E}[L1_{\{L > q_\delta(L)\}}] = \mathbb{E}[L] - \mathbb{E}[L1_{\{L \leq q_\delta(L)\}}]$  with

$$\mathbb{E}[L] = \sum_{g \in G} \sum_{k=0}^K \lambda_g w_{g,k} \mathbb{E}[L_{g,k,1}]$$

and

$$\mathbb{E}[L1_{\{L \leq q_\delta(L)\}}] = \sum_{l=1}^{q_\delta(L)} l \mathbb{P}(L = l).$$

## Contributions to Expected Shortfall – Calculation in Extended CreditRisk<sup>+</sup>

By consistency and linearity of the allocation

$$\text{ES}_\delta[L] = \text{ES}_\delta[L, L] = \sum_{i=1}^m \sum_{g \in G_i} \sum_{k=0}^K \text{ES}_\delta[L_{g,i,k}, L].$$

Since

$$\begin{aligned} \mathbb{E}[L_{g,i,k} 1_{\{L > q_\delta(L)\}}] &= \underbrace{\mathbb{E}[L_{g,i,k}]} - \mathbb{E}[L_{g,i,k} 1_{\{L \leq q_\delta(L)\}}], \\ &= \lambda_g w_{g,k} \mathbb{E}[L_{g,i,k,1}] \end{aligned}$$

we will compute  $\mathbb{E}[L_{g,i,k} 1_{\{L=l\}}]$  for  $l \in \{1, \dots, q_\delta(L)\}$ . This can be done using the following lemma.

## Contributions to Expected Shortfall – Definition

**Definition:** For a subportfolio loss  $X \in \mathcal{L}_1(\mathbb{P})$  within a portfolio loss  $Y \in \mathcal{L}_1(\mathbb{P})$  define the expected shortfall contribution at level  $\delta \in (0, 1)$  of  $X$  to  $Y$  by

$$\text{ES}_\delta[X, Y] = \frac{E[X1_{\{Y > q_\delta(Y)\}}] + \beta_Y \mathbb{E}[X1_{\{Y = q_\delta(Y)\}}]}{1 - \delta}$$

where

$$\beta_Y = \frac{\mathbb{P}(Y \leq q_\delta(Y)) - \delta}{\mathbb{P}(Y = q_\delta(Y))}$$

if  $\mathbb{P}(Y = q_\delta(Y)) > 0$  and 0 otherwise.

**Remark:** If  $\mathbb{P}(Y \leq q_\delta(Y)) = \delta$ , then  $\beta_Y = 0$  and

$$\text{ES}_\delta[X, Y] = \mathbb{E}[X | Y > q_\delta(Y)].$$

## Lemma on Risk Contributions in CreditRisk<sup>+</sup>

For every obligor  $i \in \{1, \dots, m\}$ , every group  $g \in G_i$  and total loss  $l \in \mathbb{N}_0$ ,

$$\begin{aligned} \mathbb{E}[L_{g,i,0} 1_{\{L=l\}}] &= \lambda_g w_{g,0} \sum_{\nu=1}^l \mathbb{E}[L_{g,i,0,1} 1_{\{L_{g,0,1}=\nu\}}] \mathbb{P}(L = l - \nu) \end{aligned}$$

and, for every risk  $k \in \{1, \dots, K\}$ ,

$$\begin{aligned} \mathbb{E}[L_{g,i,k} 1_{\{L=l\}}] &= \lambda_g w_{g,k} \sum_{\nu=1}^l \mathbb{E}[L_{g,i,k,1} 1_{\{L_{g,k,1}=\nu\}}] \mathbb{E}[\wedge_k 1_{\{L=l-\nu\}}]. \end{aligned}$$

## Possible Approaches to Operational Risk Modelling

The Basel committee defined three approaches towards the quantification of operational risk. The two simple ones define concrete formulae for the risk capital, namely

- Basic indicator approach (BIA).
- Standardized approach (SA).

To reduce supervisory capital needs, an individual

- Advanced measurement approach (AMA) can be chosen.

## Seven Loss-Types Distinguished for the Advanced Measurement Approach

- Internal fraud
- External fraud
- Employment practices & workplace safety
- Clients, products & business practice
- Damage to physical assets
- Business disruption & system failures
- Execution, delivery & process management

## Business Lines for Operational Risk

- Eight business lines in the standardized approach:
  - (1) Corporate finance,
  - (2) Trading & sales,
  - (3) Retail banking,
  - (4) Commercial banking,
  - (5) Payment & settlement,
  - (6) Agency services,
  - (7) Asset management,
  - (8) Retail brokerage.
- These business lines also serve as categories for an advanced measurement approach.

## Application of Extended CreditRisk<sup>+</sup> Methodology to Operational Risk: Reinterpretation of the Credit Risk Notation

- Number  $m$  of obligors  $\rightarrow$  number of business lines ( $m = 8$  for the ones given in the Basel committee's document is an appropriate choice).
- Basic loss unit  $E$  stays the same ( $E = 10\,000$ ).
- Number  $K$  of non-ideosyncratic risk factors  $\rightarrow$  number of loss types ( $K = 7$  for above list).
- Numbers  $\sigma_k^2 > 0$  denote the relative variance of occurrences of losses of type  $k \in \{1, \dots, K\}$ .
- $G$  contains the subsets of all business lines which can incur a loss due to the same event.

## Notation for Business Lines and Risk Groups

We need for every risk group  $g \in G$  of business lines

- the (one year) intensity  $\lambda_g \geq 0$  for being hit by an operational loss event,
- the conditional probability  $w_{g,0} \in [0, 1]$  for an idiosyncratic operational loss event not to belong to the types in  $\{1, \dots, K\}$ , of course  $w_{g,0} = 0$  is a possible choice,
- the conditional probabilities  $w_{g,k} \in [0, 1]$  for an operational loss event to be of type  $k \in \{1, \dots, K\}$ ,

## Further Extensions of CreditRisk<sup>+</sup>

- Dependent risk factors, possibly with an interactive and hierarchical structure
- Other mixture distributions besides the gamma distribution
- Special choices of weighted expected shortfall as risk measure and corresponding risk contribution

- the multivariate probability distribution  $Q_{g,k} = \{q_{g,k,\mu}\}_{\mu \in \mathbb{N}_0^g}$  on  $\mathbb{N}_0^g$  describing the severity of the stochastic losses of the business lines  $i \in g$  in case an operational loss event of type  $k \in \{0, \dots, K\}$  hits the group  $g$  of business lines.

## Operational Risk Management

With the adoption of the extended CreditRisk<sup>+</sup> model for operational risk, a risk manager can

- calculate the distribution of the operational loss,
- calculate risk measures such as value-at-risk and expected shortfall (might be infinity)
- and identify risky business lines and groups by their risk contribution (in case of finite expected shortfall).