Introduction to Risk Measures and Capital Allocation Principles

Introductory Crash Course
Austrian Workshop on ALM in Insurance
September 23–25, 2004, Vienna

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Purpose of Risk Measures

• Concentrate the “relevant” information about the future worth of a risky position into a single number.

• Determine the amount of cash (or units of a reference instrument) needed to make a risky position acceptable for the period.

Remarks

• Connection to premium calculation principles

• Risk measures should have economically meaningful properties, in particular w.r.t. aggregation of risks.

• Focus on the loss part (one-sided measures); variance and standard deviation punish free lottery tickets.
Quantiles and Value-at-Risk

\[ X : \Omega \rightarrow \mathbb{R} \text{ random discounted one-period profit & loss} \]

- Upper \( \alpha \)-quantile of \( X \) with \( \alpha \in (0, 1) \)
  \[ q^\alpha(X) = \inf\{ x \in \mathbb{R} \mid P(X \leq x) > \alpha \} \]

- Lower \( \alpha \)-quantile of \( X \)
  \[ q_\alpha(X) = \inf\{ x \in \mathbb{R} \mid P(X \leq x) \geq \alpha \} \]

\( q_\alpha(X) = q^\alpha(X) \iff P(X \leq x) = \alpha \text{ for at most one } x \)

Value-at-Risk of \( X \) at level \( \alpha \)
\[ \text{VaR}_\alpha(X) = -q^\alpha(X) = q_{1-\alpha}(-X) \]

Smallest value when added to \( X \) avoids negative results with probability at least \( 1 - \alpha \).

Advantages and Deficiencies of Value-at-Risk

- Robust quantity like the median, doesn’t depend of the far-out tails.
- “Easy” to calculate and to backtest.
- Applicable for all real-valued random variables.

Deficiencies:

- \( \text{VaR} \) ignores severity of unfavourable events.
- \textbf{VaR can punish diversification!} Example:
  \( (a) \) 100 Euro loan with default probability \( p = 0.8\% \)
  \[ \Rightarrow \text{VaR}_{1\%}(X) = 0 \]
  \( (b) \) Two independent 50 Euro loans with \( p = 0.8\% \)
  \[ \Rightarrow P(\text{at least one default}) = 2p - p^2 \geq 1.59\% \]
  Therefore \( \text{VaR}_{1\%}(X) = 50 \)
**Definition of a Coherent Risk Measure** (ADEH 1999)

A map $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, defined on the set of $(\mathbb{P}$-equivalence classes of) $\mathbb{P}$-almost surely bounded random variables on $(\Omega, \mathcal{F})$, is called coherent risk measure if it satisfies

(a) Monotonicity: $X \geq 0 \implies \varrho(X) \leq 0$,
(b) Positive homogeneity: $\varrho(\lambda X) = \lambda \varrho(X)$ for all $\lambda \geq 0$,
(c) Translation invariance: $\varrho(X + c) = \varrho(X) - c$ for all $c \in \mathbb{R}$,
(d) Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.

VaR satisfies conditions (a)–(c), but not (d).

**Examples of Coherent Risk Measures**

Let $\mathcal{P}$ be a set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, absolutely continuous w.r.t. $\mathbb{P}$ (think of scenarios). Then a coherent risk measure $\varrho_{\mathcal{P}} : L^\infty \to \mathbb{R}$ is given by

$$\varrho_{\mathcal{P}}(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-X], \quad X \in L^\infty.$$

- If $\mathcal{P} = \{\mathbb{P}\}$, then $\varrho_{\mathcal{P}}(X) = \mathbb{E}_{\mathbb{P}}[-X]$ (too tolerant).
- If $\mathcal{P} = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}\}$, then $\varrho_{\mathcal{P}}(X) = \text{ess sup}_{\mathbb{P}}(-X)$ (too restrictive).
- For $\alpha \in (0, 1)$ define $\mathcal{P}_\alpha = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\}$. 
**Tail Mean and Expected Shortfall**

For measurable $X : \Omega \rightarrow \mathbb{R}$ with $\mathbb{E}[X^-] < \infty$ define the tail mean of $X$ at level $\alpha \in (0, 1)$ by

$$TM_{\alpha}(X) = \frac{1}{\alpha} \mathbb{E}[X 1\{X < q^\alpha(X)\}] + q^\alpha(X) \frac{\mathbb{P}(X < q^\alpha(X))}{\alpha}.$$ 

If $\mathbb{P}(X \leq q^\alpha(X)) = \alpha$, then

$$TM_{\alpha}(X) = \mathbb{E}[X | X \leq q^\alpha(X)].$$

Define the expected shortfall of $X$ at level $\alpha \in (0, 1)$ by

$$ES_{\alpha}(X) = -TM_{\alpha}(X) \geq -q^\alpha(X) = \text{VaR}_\alpha(X).$$

**Theorem:** $\varrho_{\mathcal{P}}(X) = ES_{\alpha}(X)$

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**Characterization of Coherent Risk Measures**

**Definition:** A coherent risk measure $\varrho$ is said to satisfy the Fatou property, if for every $X$ and every sequence $\{X_n\}_{n \in \mathbb{N}}$ in $L^\infty$ with $\|X_n\|_\infty \leq 1$

$$X_n \overset{p}{\rightarrow} X \implies \varrho(X) \leq \liminf_{n \rightarrow \infty} \varrho(X_n).$$

**Theorem:** A coherent risk measure $\varrho : L^\infty \rightarrow \mathbb{R}$ satisfies the Fatou property if and only if there exists an $L^1(\mathbb{P})$-closed, convex set of probability measures $\mathcal{P}$ with $Q \ll \mathbb{P}$ for all $Q \in \mathcal{P}$ such that

$$\varrho(X) = \varrho_{\mathcal{P}}(X) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[-X], \quad X \in L^\infty.$$
Extension of a Coherent Risk Measure

Let $\varrho_P : L^\infty \to \mathbb{R}$ be a coherent risk measure as in the previous theorem. Suppose the exists $\delta > 0$ such that

$$A \in \mathcal{F} \text{ satisfies } P(A) \leq \delta \implies \text{ there exists } Q \in \mathcal{P} \text{ with } Q(A) = 0.$$ 

Then

$$\varrho(X) = \lim_{n \to \infty} \sup_{Q \in \mathcal{P}} E_Q[-(X \wedge n)]$$

defines an extension $\varrho : L^0 \to \mathbb{R} \cup \{\infty\}$ of $\varrho_P$ to the space $L^0(\Omega, \mathcal{F}, P)$ of all random variables preserving monotonicity, positive homogeneity, translation invariance and subadditivity.

Convex Risk Measures (Föllmer/Schied 2000)

Risk for large positions might increase more than linear (due to additional liquidity risk, for example), hence positive homogeneity might need to be relaxed.

Definition: A map $\varrho : L^\infty \to \mathbb{R}$ is called convex risk measure if it satisfies

(a) Monotonicity: $X \geq Y \implies \varrho(X) \leq \varrho(Y)$,
(b) Convexity: For all $\lambda \in [0, 1]$
$$\varrho(\lambda X + (1 - \lambda)Y) \leq \lambda \varrho(X) + (1 - \lambda)\varrho(Y),$$
(c) Translation invariance: For all $c \in \mathbb{R}$
$$\varrho(X + c) = \varrho(X) - c.$$
Convex Risk Measure Defined by a Loss Function

- Loss funct. $l : \mathbb{R} \to \mathbb{R}$ increasing, convex, non-constant
- Threshold $x_0$ in the range of $l$

Then a convex risk measure $\varrho_l : L^\infty \to \mathbb{R}$ is defined by

$$\varrho_l(X) = \inf \{ c \in \mathbb{R} \mid \mathbb{E}[l(-(X + c))] \leq x_0 \}.$$ 

Examples:

- $l(x) = \exp(\lambda x)$ with $\lambda > 0$ and $x_0 = 1$
  $$\implies \varrho_l(X) = \frac{1}{\lambda} \log \mathbb{E}[\exp(-\lambda X)]$$
- $l(x) = \max\{x, 0\}$ and $x_0 \geq 0$
  $$\implies \varrho_l(X)$$ is minimal retention level such that the expected excess of loss is bounded by $x_0$.
- $l(x) = \frac{1}{p}(\max\{x, 0\})^p$ with $p \geq 1$

Characterization of Convex Risk Measures

**Definition:** A convex risk measure $\varrho$ is said to satisfy the Fatou property, if for every $X$ and every sequence $\{X_n\}_{n \in \mathbb{N}}$ in $L^\infty$ with $\|X_n\|_\infty \leq 1$

$$X_n \overset{p}{\to} X \implies \varrho(X) \leq \liminf_{n \to \infty} \varrho(X_n)$$

**Theorem:** A convex risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfies the Fatou property if and only if there exists a “penalty function” $\alpha : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ such that

$$\varrho(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \left( \mathbb{E}_\mathbb{Q}[-X] - \alpha(\mathbb{Q}) \right), \quad X \in L^\infty,$$

where $\mathcal{P}$ is the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous w.r.t. $\mathbb{P}$. 

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The Allocation Problem for Risk Capital

Given risk bearing capital $C > 0$ for a financial institution, how to allocate it to business units for

- Fair distribution of the diversification benefit,
- Consideration of dependencies (ALM),
- Measurement of risk contributions (risk management),
- Performance measurement (for steering the company),
- Determination of bonuses for the management?

Applications on the Portfolio Level

- Security loadings for individual insurance contracts
- Credit spreads for loans and defaultable bonds

A Wish List for the Allocation of Risk Capital

- Coherent risk measure $\varrho: L^\infty \rightarrow \mathbb{R}$
- Profit & losses $X_1, \ldots, X_m$ of $m$ business units, adding up to the company result $X = X_1 + \cdots + X_m$
- Total risk capital $C$, capital $C_i$ assigned to unit $i$

Useful Properties (cf. game theory)

1. Risk sensitivity: $C = \varrho(X)$
2. Additivity: $C = C_1 + \cdots + C_m$
3. No subgroup of units is better off on its own:
   $\sum_{i \in I} C_i \leq \varrho(\sum_{i \in I} X_i)$ for all $I \subset \{1, \ldots, m\}$.
4. If business units can be divided into parts:
   $\sum_{i=1}^m \alpha_i C_i \leq \varrho(\sum_{i=1}^m \alpha_i X_i)$ for all $\alpha_i \in [0, 1]$. 
Axiomatic Approach to Risk Capital Allocation

Let $V \subset L^0$ be a linear subspace, e.g. $V = L^\infty$.

**Def.:** $\Lambda : V \times V \to \mathbb{R}$ is called a risk capital allocation principle w.r.t. the coherent risk measure $\varrho : V \to \mathbb{R}$, if it satisfies for all $X, Y, Z \in V$ and $\alpha, \beta \in \mathbb{R}$

(a) **Risk sensitivity:** $\Lambda(X, X) = \varrho(X)$,

(b) **Linearity:** $\Lambda(\alpha X + \beta Y, Z) = \alpha \Lambda(X, Z) + \beta \Lambda(Y, Z)$,

(c) **Diversification:** $\Lambda(X, Y) \leq \Lambda(X, X)$.

**Exercise:** Such a $\Lambda$ has the four useful properties.

$\Lambda$ is called continuous at $Y \in V$, if for all $X \in V$

$$\lim_{\varepsilon \to 0} \Lambda(X, Y + \varepsilon X) = \Lambda(X, Y).$$

Results About Risk Capital Allocations

**Existence:** For every $Y \in V$ there exists $h_Y \in V^*$ with $h_Y(Y) = \varrho(Y)$ and $h_Y \leq \varrho$ on $V$. Furthermore,

$$\Lambda(X, Y) = h_Y(X), \quad X, Y \in V,$$

defines a risk capital allocation principle w.r.t. the coherent risk measure $\varrho$.

**Uniqueness:** If the capital allocation principle $\Lambda$ is continuous at $Y \in V$, then for all $X \in V$

$$\Lambda(X, Y) = \lim_{\varepsilon \to 0} \frac{\varrho(Y + \varepsilon X) - \varrho(Y)}{\varepsilon}$$

(directional derivative of the underlying risk measure).
Consider $\alpha \in (0, 1)$ and $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. A capital allocation principle for $\varrho_{P_\alpha} = ES_\alpha$ is

$$
\Lambda^{ES}_\alpha (X, Y) = -\frac{1}{\alpha} \mathbb{E}[X 1\{Y < q_\alpha(Y)\}] \\
- \frac{\alpha - \mathbb{P}(Y < q_\alpha(Y))}{\alpha} \mathbb{E}[X | Y = q_\alpha(Y)].
$$

If $\mathbb{P}(Y \leq q_\alpha(Y)) = \alpha$, then

$$
\Lambda^{ES}_\alpha (X, Y) = - \mathbb{E}[X | Y \leq q_\alpha(Y)].
$$

If $\mathbb{P}(Y = q_\alpha(Y)) = 0$, then $\Lambda^{ES}_\alpha$ is continuous at $Y$.

Further Topics

- Risk measures and acceptance sets
- Risk measures and utility functions
- Value of information and risk measures
- Convex risk measures and convex trading constraints
- Allocation of risk capital and game theory
- Multi-period risk measurement
Some Literature Related to Risk Measures