A Generalization of the Collective Theory of Risk in Regard to Fluctuating Basic-Probabilities.

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Generally in the theory of risk one starts from the two fundamental assumptions that the basic-probabilities are constant and that the deviations occurring may be interpreted as random fluctuations. Thus, the theory of risk appears as an application of the ordinary probability-theory, which starts from the binomial distribution and leads to the distributions of Poisson and Gauss.

In certain insurance-branches these assumptions agree more or less with the reality, so that the application of the ordinary probability-theory seems to be justified in these insurance-branches, at least as a first approximation. But as a rule the application of the ordinary probability-theory must be considered as doubtful, because the fluctuations occurring do not follow the above mentioned distributions of Poisson and Gauss. From these circumstances many actuaries draw the conclusion that the ordinary probability-theory is not applicable to insurance-problems and that the practical application of the theory of risk is not justified.

From our point of view this opinion does not hit the essential point; not the probability-theory in itself is inapplicable to insurance-problems, but its results, as long as they are based on too simple assumptions. In particular the assumption of constant basic-probabilities might only agree insufficiently with the real facts in the insurance field. On the other hand it seems quite possible to conceive a probability- and risk-theory without this special assumption. Cramér has already pointed out in his paper "On the mathematical theory of risk", that the theory of risk might also be based on fluctuating basic-probabilities. Further Nolfi has developed in his paper of the Congress of 1940 an individual theory of risk based on the assumption of fluctuating basic-probabilities.

In this paper we will try to introduce the same ideas into the collective theory of risk. First using a generalized urn-model the theory will be developed in an elementary manner. Then we shall proceed to a more detailed discussion of the probability-function of the gain and to some remarks on the so-called "ruin-problem".

I. A Generalized Urn-model.

1) In the classical urn-model of Bernoulli, we draw from a single urn, containing R red and $S = N - R$ black balls, ν balls at random, whereby the balls drawn will always be replaced in the urn. The probability $f_1(\varrho, \nu)$ of drawing ϱ red balls we obtain from the binomial distribution

$$f_1(\varrho, \nu) = \binom{\nu}{\varrho} \left(\frac{R}{N}\right)^{\varrho} \left(\frac{S}{N}\right)^{\nu-\varrho} \tag{1}$$

By means of certain limiting processes we deduce from the binomial distribution the distributions of Poisson and Gauss.

2) We now generalize the classical urn-model by replacing the single urn by a series of K urns, whereby the proportion between red and black balls varies from urn to urn, e. g. there are R_k red and $S_k = N - R_k$ black balls in the k^{th} urn. Besides these K urns, which we call sub-urns, we have a so-called main-urn, which contains $H \geq K$ tickets, numbered from one to K, each ticket corresponding to a certain sub-urn; there may be several tickets of the same number in the main-urn, e. g. there are h_k tickets with the number k. We have then the equation $\sum_{k=1}^{K} h_k = H$. As in the classical urn-model of Bernoulli we draw ν balls from one and the same sub-urn, whereby the sub-urn in question has been chosen by drawing a ticket from the main-urn. The probability of drawing ϱ red balls we obtain from the frequency-function

$$f_1(\varrho, \nu) = \sum_{k=1}^{K} \frac{h_k}{H} \binom{\nu}{\varrho} \left(\frac{R_k}{N}\right)^{\varrho} \left(\frac{S_k}{N}\right)^{\nu-\varrho}, \tag{2}$$

to which corresponds the characteristic function

$$_{\varrho}\varphi_1(t, \nu) = \sum_{\varrho=0}^{\nu} f_1(\varrho, \nu)\, e^{it\varrho} = \sum_{k=1}^{K} \frac{h_k}{H} \left(\frac{e^{it} R_k + S_k}{N}\right)^{\nu}. \tag{2 a}$$

Both the frequency-function and the characteristic function are weighted means of the corresponding frequency-functions or characteristic functions in all sub-urns.

From the characteristic function we obtain the moments m_r of the frequency-function by means of the formula

$$m_r = i^{(-r)}\, \varphi^{(r)}(0), \tag{3}$$

in which $\varphi^{(r)}(0)$ denotes the r^{th} derivative of the characteristic function for $t = 0$. In particular we have for the two first moments about the origin of the frequency-function (2)

$$m_1 = \bar{\varrho} = \frac{\nu}{N} \sum_{k=1}^{K} \frac{h_k R_k}{H}$$

and

$$m_2 = \sum_{k=1}^{K} \frac{h_k}{H} \left\{ \frac{\nu^2 R_k^2}{N^2} - \frac{\nu R_k^2}{N^2} + \frac{\nu R_k}{N} \right\}.$$

For the second moment about the mean or the variance we further have

$$m_2' = m_2 - \bar{\varrho}^2 = \sigma^2\left(\frac{R_k}{N}\right) \nu(\nu - 1) + \nu \left(\frac{\bar{\varrho}}{\nu}\right)\left(1 - \frac{\bar{\varrho}}{\nu}\right), \tag{4}$$

in which we have introduced the variance of the quantities $\left(\frac{R_k}{N}\right)$, which we will call the main-variance

$$\sigma^2\left(\frac{R_k}{N}\right) = \sum_{k=1}^{K} \frac{h_k}{H} \left\{\frac{R_k}{N} - \frac{\bar{\varrho}}{\nu}\right\}^2. \tag{4 a}$$

Formula (4) shows that the variance of the frequency-function (2) consists of two parts, the first depending on the main-variance (4 a) and the second identical with the variance of the binomial distribution (1). Thus the variance of the frequency-function (2), based on the generalized urn-model, is larger than the variance of the binomial distribution, based on the classical urn-model of Bernoulli.

3) The formulas deduced above refer to the case of a single series of ν drawings from a single sub-urn, which has been chosen by means of a ticket drawn from the main-urn. Now we study the case of a sequence of n series of ν drawings, whereby at the beginning of each serie the sub-urn in question will be chosen by means of the main-urn. In order to be quite clear we call the quantity ν in the following "the length" of the series. In the classical urn-model we arrive at the same result, whether we consider a single series of the length $n\nu$, or a sequence of n series with the length ν. In our generalized urn-model we obtain, however, new distributions, which on account of the mutual independence of the different series follow from the recurrence formula

$$f_n(\varrho, \nu) = \sum_{\varrho'=0}^{\varrho} f_1(\varrho', \nu) f_{n-1}((\varrho - \varrho'), \nu). \tag{5}$$

A simpler equation we get for the corresponding characteristic functions; by means of the product-theorem we obtain

$$_{\varrho}\varphi_n(t, \nu) = [_{\varrho}\varphi_1(t, \nu)]^n \tag{5 a}$$

a formula, which may be generalized without any difficulties to the case of series with different lengths

$$_{\varrho}\varphi_n(t, \nu_1, \nu_2, \ldots \nu_n) = \prod_{r=1}^{n} [_{\varrho}\varphi_1(t, \nu_r)]. \tag{5 a'}$$

4) Supposing that the quantity K of the sub-urns and of the corresponding tickets in the main-urn becomes larger and larger, so that finally the ratios of combination in the sub-urns varie continuously from one sub-urn to the other, the discontinuous frequency-function $h\left(\frac{R_k}{N}\right) = \frac{h_k}{H}$ changes into the continuous frequency-function $h(p)$; $h(p)\,dp$ denotes the probability of choosing at random a sub-urn, in which we have a probability in the range between p and $p + dp$ of getting a red ball out of a single drawing. By means of the continuous distribution $h(p)$ — we call it the main-distribution — we obtain the following formulas in the place of the formulas (2) and (2 a)

$$f_1(\varrho, \nu) = \int_0^1 \binom{\nu}{\varrho} p^\varrho (1-p)^{\nu-\varrho} h(p)\,dp \tag{2'}$$

$$_\varrho\varphi_1(t, \nu) = \int_0^1 \{(1-p) + p\,e^{it}\}^\nu h(p)\,dp. \tag{2 a'}$$

In the following we always assume a continuous main-distribution.

5) In all sub-urns we now let tend the probability p to zero and at the same time the length of the series ν to infinity, so that the mean $\nu p = P$ remains constant; in this way we get in all the sub-urns the distribution of Poisson

$$f_1(\varrho, P) = \frac{e^{-P} P^\varrho}{\varrho!} \tag{6}$$

in the place of the binomial distribution. Further we introduce a transformed main-distribution $h(q)$ instead of $h(p)$; $h(q)\,dq$ means the probability of choosing at random a sub-urn in which the proportion between red and black balls is such, that the mean of series of drawings is within the range of qP and $(q + dq)\,P$. To the limiting case of Poisson correspond in our generalized urn-model the formulas:

$$f_1(\varrho, P) = \int_0^\infty \frac{e^{-Pq}(Pq)^\varrho}{\varrho!} h(q)\,dq \tag{2''}$$

and

$$_\varrho\varphi_1(t, P) = \int_0^\infty e^{Pq(e^{it}-1)} h(q)\,dq. \tag{2 a''}$$

6) From the ordinary probability-theory we know that the binomial as well as the Poisson distribution for large values of ν or P tend to the normal distribution of Gauss-Laplace. In order to derive the limiting distribution in our generalized urn-model, we introduce the auxiliary-variable

$$u = \frac{\varrho - P}{\sigma(\varrho)},$$

whereby $\sigma(\varrho)$ means the standard deviation of the frequency-function (2″)

$$\sigma(\varrho) = P\sigma(q)\sqrt{1 + P^{-1}\sigma^{-2}(q)}\,.$$

Substituting now the variable u for ϱ we get by means of the well known rules for linear substitutions in the characteristic functions

$$_u\varphi_1(t, P) = e^{-\frac{itP}{P\sigma(q)\sqrt{1+P^{-1}\sigma^{-2}(q)}}} \int_0^\infty e^{Pq\left(e^{\frac{it}{P\sigma(q)\sqrt{1+P^{-1}\sigma^{-2}(q)}}}-1\right)} h(q)\,dq =$$

$$= \int_0^\infty e^{-\frac{it}{\sigma(q)\sqrt{1+P^{-1}\sigma^{-2}(q)}}} e^{Pq\left\{\frac{it}{P\sigma(q)\sqrt{1+P^{-1}\sigma^{-2}(q)}} + \frac{(it)^2}{2!}\cdots\right\}} h(q)\,dq\,.$$

In the limit $P \to \infty$ we must distinguish between the two cases $\sigma(q) \neq 0$ and $\sigma(q) = 0$, and we obtain

$$\lim_{P\to\infty} {}_u\varphi_1(t, P) = \int_0^\infty e^{\frac{-it}{\sigma(q)}} e^{\frac{itq}{\sigma(q)}} h(q)\,dq \qquad \sigma(q) \neq 0 \qquad (7\text{ a})$$

$$\lim_{P\to\infty} {}_u\varphi_1(t, P) = e^{\frac{-t^2}{2}}, \qquad \sigma(q) = 0 \qquad (7\text{ b})$$

uniformly in any finite t-interval.

The last mentioned case (7 b) corresponds to the well known result in the ordinary probability-theory, appearing here as a special case. Replacing in formula (7 a) the variable u by the variable

$$q = \frac{\varrho}{P} = 1 + u\,\sigma(q),$$

we arrive at

$$\lim_{P\to\infty} {}_0\varphi_1(t, P) = \int_0^\infty e^{itq}\, h(q)\, dq, \qquad \sigma(q) \neq 0 \qquad (7\,a')$$

which is the characteristic function corresponding to the main-distribution $h(q)$. On account of the uniform convergence we conclude from (7 a′) that the main-distribution $h(q)$ is identical with our limiting distribution for large values of P. Thus the analogous limiting process, which leads in the classical urn-model to the normal distribution, leads in our generalized urn-model to quite a different result.

II. The Probability-Function of the Gain in the Collective Theory of Risk.

a) *General Assumptions.*

7) In the following we consider exclusively the risk business of an insurance-institution. The insurance stock of this institution is composed in such a manner, that we have a probability of $p(z)\,dz$ that any sum at risk due for payment falls in the range between z and $z + dz$. As usual the frequency-function $p(z)$ may satisfy the equations

$$\int_0^\infty p(z)\,dz = \int_0^\infty z\,p(z)\,dz = 1,$$

i. e. we suppose for the sake of simplicity that only positive sums at risk occur, and that the mean sum at risk will be taken as unit of calculation. Further we assume, that $p(z)$ is continuous for all values of z and invariable during the time, and finally that the integral

$$\int_0^\infty e^{Rz}\,p(z)\,dz$$

converges for values of $R < R_0$.

The extent of the risk business we suppose to be constant in the sense that the expected number of claims P during a certain unit-period of observation remains constant, as long as we assume constant basic-probabilities, or fluctuates about a constant mean-value, as long as we assume fluctuating basic-probabilities. The fluctuations of P, which may be caused in practice by the change of certain political, economical or climatic circumstances, follow a main-distribution. From the frequency-function of this main-distribution we obtain the probability $h(q)dq$ that the expected number of claims in a unit-period falls within the range between Pq and $P(q+dq)$. In our following investigations we renounce the possibility of basing the theory on a general main-distribution; we assume the Pearson-curve type III

$$h(q) = \frac{h_0^{h_0} e^{-h_0 q} q^{h_0 - 1}}{\Gamma(h_0)} \tag{8}$$

to be the frequency-function of the main-distribution, in which h_0 is a parameter measuring the precision of the distribution. We have chosen this special distribution on account of its analytical qualities, and because it agrees with certain general conditions to be expected from each main-distribution, namely that its range of definition is from zero to infinity, that its mean is

$$\int_0^\infty q\, h(q)\, dq = 1,$$

and finally that it contains the assumption of constant basic-probabilities with $h_0 = \infty$ as a particular case. Special attention must be given to the fact, that the main-distribution is always defined in connection with a certain unit-period of observation. In practical applications the calendar-year will as a rule be taken as unit-period.

The quantity P, the mean value of the expected number of claims in a unit-period, is, on account of the unit of calculation chosen, identical with the total amount of net-risk-premiums belonging to the unit-period in question.

In the following we discuss in a somewhat more detailed manner the so-called probability-function $F(g, P)$ of the gain

g for a unit-period, such a function denoting the probability of realizing a gain $g' \leq g$ during a unit-period.

Our further investigations are based on the generalized urn-model described above, in particular on Poisson's limiting case treated under 5). Indeed we may easily interpret the frequency-function (2″) as the frequency-function of the total amount to be paid out in a unit-period for the special case of uniform sums at risk. The general case with different sums at risk we easily obtain from the generalized urn-model, if we assume that each red ball drawn has a certain value differing from ball to ball.

b) *Some Investigations about the Probability Function of the Gain.*

8) Let us consider at first the frequency-function of the total amount x to be paid out in a unit-period. We denote this frequency-function with ${}^{(h_0)}f_1(x, P)$ or ${}^{(\infty)}f_1(x, P)$ according as fluctuating or constant basic-probabilities are taken into consideration. These two frequency-functions depend on one another by the equation

$$ {}^{(h_0)}f_1(x, P) = \int_0^\infty {}^{(\infty)}f_1(x, Pq)\, h(q)\, dq. \tag{9} $$

For the corresponding characteristic functions we further have

$$ {}^{(h_0)}_{\;\;x}\varphi_1(t, P) = \int_0^\infty {}^{(\infty)}_{\;\;x}\varphi_1(t, Pq)\, h(q)\, dq $$

$$ = \frac{1}{\Gamma(h_0)} \int_0^\infty h_0^{h_0} e^{-h_0 q} q^{h_0-1} e^{Pq \int_0^\infty (e^{itz}-1)\, p(z)\, dz} dq\,. $$

Introducing the characteristic function of $p(z)$

$$ {}_z\pi(t) = \int_0^\infty e^{itz} p(z)\, dz\,, $$

we finally get after some reduction

$$ {}^{(h_0)}_{\;x}\varphi_1(t, P) = \left\{ 1 - \frac{P}{h_0}\left({}_z\pi(t) - 1\right) \right\}^{-h_0}. \tag{10} $$

For $h_0 \to \infty$ formula (10) changes into the well known expression

$$ {}^{(\infty)}_{\;x}\varphi_1(t, P) = e^{P({}_z\pi(t) - 1)}, \tag{10'} $$

which holds true for constant basic-probabilities.

The gain g we obtain from the quantity x by the simple substitution

$$ g = P - x; $$

thus we have for the characteristic function corresponding to the probability-function of the gain

$$ {}^{(h_0)}_{\;g}\varphi_1(t, P) = e^{itP} \left\{ 1 - \frac{P}{h_0}\left({}_z\pi(-t) - 1\right) \right\}^{-h_0}. \tag{10 a} $$

The probability-function itself we get by means of the inversion-theorem

$$ {}^{(h_0)}F_1(g, P) = {}^{(h_0)}F_1(0, P) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 - e^{itg}}{t} \, {}^{(h_0)}_{\;g}\varphi_1(t, P)\, dt. \tag{11} $$

9) The formulas (10) and (11) give all necessary information about the probability-function of the gain of a single unit-period, e.g. a calendar-year. The corresponding probability-function for n unit-periods we calculate assuming the total amount x to be paid out in different unit-periods to be mutually independent in the stochastical sense. By means of the product-theorem we get for the characteristic functions

$$ \begin{aligned} {}^{(h_0)}_{\;x}\varphi_n(t, P) &= \left\{ {}^{(h_0)}_{\;x}\varphi_1(t, P) \right\}^n \\ &= \left\{ 1 - \frac{P}{h_0}\left({}_z\pi(t) - 1\right) \right\}^{-n h_0} \\ &= {}^{(n h_0)}_{\;\;x}\varphi_1(t, n P). \end{aligned} \tag{12} $$

Formula (12) shows, that the characteristic function for a period of the length n is constructed in the same manner as the characteristic function for the unit-period, only P and h_0 change into nP and nh_0. We obviously have also the same relation for the corresponding probability-functions. In the following we may therefore deal with the case of a single unit-period only.

From formula (12) we can easily deduce a further result. If we assume for a moment that the unit-period changes from one to $\frac{1}{n}$ without changing the value of h_0, we have

$$ {}^{(h_0)}_{\;x}\varphi_{\frac{1}{n}}\left(t, \frac{P}{n}\right) = \left\{1 - \frac{P}{n \cdot h_0}\left({}_z\pi(t) - 1\right)\right\}^{-h_0} $$

and in the limit $n \to \infty$ for $n \cdot \frac{1}{n}$ unit-periods

$$ \lim_{n \to \infty} {}^{(h_0)}_{\;x}\varphi_{\frac{n}{n}}\left(t, \frac{P}{n}\right) = e^{P({}_z\pi(t) - 1)}. \tag{13} $$

The expression on the right hand side of formula (13) is identical with the characteristic function derived for constant basic-probabilities (formula 10′). Thus basic-probabilities fluctuating from moment to moment lead to the same probability-functions as constant basic-probabilities.

10) Important measures of the probability-functions are their moments, which we get in differentiating the characteristic function (formula (3)). For the moments about the origin of the frequency-function of x we get

$$ \left.\begin{aligned} M_0 &= 1 \\ M_1 &= P \\ M_2 &= Pp_2 + \frac{P^2}{h_0^2}\frac{\Gamma(h_0+2)}{\Gamma(h_0)} \\ M_3 &= Pp_3 + \frac{3P^2}{h_0^2}p_2\frac{\Gamma(h_0+2)}{\Gamma(h_0)} + \frac{P^3}{h_0^3}\frac{\Gamma(h_0+3)}{\Gamma(h_0)}, \end{aligned}\right\} \tag{14} $$

in which we have introduced the moments about the origin of the frequency-function $p(z)$ by $p_1 = 1, p_2, p_3 \ldots$. From these moments M_r about the origin we arrive at the moments M'_r about the mean by using the equations

$$\left.\begin{aligned} M'_1 &= 0 \\ M'_2 &= M_2 - M_1^2 \\ M'_3 &= M_3 - 3\,M_2 M_1 + 2\,M_1^3 \end{aligned}\right\} \tag{15}$$

to

$$\left.\begin{aligned} M'_1 &= 0 \\ M'_2 &= P p_2 + P^2\,h_0^{-1} \\ M'_3 &= P p_3 + 3\,P^2 p_2\,h_0^{-1} + 2\,P^3\,h_0^{-2}. \end{aligned}\right\} \tag{16}$$

Putting $\dfrac{P}{h_0} = \chi$ and

$$\begin{aligned} p_2^* &= p_2 + \chi \\ p_3^* &= p_3 + 3\,p_2\,\chi + 2\,\chi^2, \end{aligned}$$

we get the clearer formulas

$$\left.\begin{aligned} M'_1 &= 0 \\ M'_2 &= P p_2^* \\ M'_3 &= P p_3^*. \end{aligned}\right\} \tag{16'}$$

For the moments M''_r of the frequency-function of the gain g we finally have

$$M''_r = (-1)^r\,M'_r,$$

whereby M'_r must be taken from the formulas (16) or from the formulas (16′).

11) A closer examination of the formulas (16′) and (16) shows that the moments (16) for $h_0 = \infty$, i. e. for constant basic-probabilities, are constructed in the same manner as the moments (16′), which take into account of fluctuating basic-probabilities; only the quantities $p_2, p_3 \ldots$ change into p_2^*, $p_3^* \ldots$. This statement leads us to the conjecture, that the probability-function of the gain, which is based on fluctuating basic-probabilities, may be deduced from the corresponding probability-function based on constant basic-probabilities by means of a certain transformation of the frequency-function $p(z)$. In order to get this transformation we put

$$e^{P^*({}_z\pi^*(t)-1)} = \left\{1 - \frac{P}{h_0}({}_z\pi(t) - 1)\right\}^{-h_0}$$

$$= (1+\chi)^{-\frac{P}{\chi}}\left(1 - \frac{\chi}{\chi+1}\,{}_z\pi(t)\right)^{-\frac{P}{\chi}},$$

whereby ${}_z\pi^*(t)$ denotes the characteristic function of the transformed frequency-function $p^*(z)$, and P^* the total amount of net-risk-premiums belonging to $p^*(z)$ and to a unit-period. Introducing logarithms we further have

$$P^*[{}_z\pi^*(t) - 1] = -\frac{P}{\chi}\left\{ln(1+\chi) + ln\left(1 - \frac{\chi}{\chi+1}\,{}_z\pi(t)\right)\right\}.$$

On the right hand side we develop the second term in a convergent series

$$P^*[{}_z\pi^*(t) - 1] = -\frac{P}{\chi}\left\{ln(1+\chi) - \sum_{r=1}^{\infty}\left(\frac{\chi}{\chi+1}\right)^r \frac{{}_z\pi^r(t)}{r}\right\}.$$

If we further consider, that the r^{th} power of the characteristic function ${}_z\pi(t)$ is equal to the characteristic function of the frequency-function $p^{(r)}(z)$, from which we deduce the probability $p^{(r)}(z)\,dz$ of paying out a total amount between z and $z + dz$ for r claims, and which follows the recurrence formula

$$p^{(r)}(z) = \int_0^{\infty} p(z_1)\,p^{(r-1)}(z - z_1)\,dz_1 = p(z) * p^{(r-1)}(z),$$

we finally obtain

$$p^*(z) = \frac{1}{ln(1+\chi)}\sum_{r=1}^{\infty}\left(\frac{\chi}{\chi+1}\right)^r \frac{p^{(r)}(z)}{r}$$

and (17)

$$P^* = \frac{P\,ln(1+\chi)}{\chi}.$$

Thus the transformed frequency-function $p^*(z)$ presents itself as a linear combination of all the frequency-functions $p^{(r)}(z)$, resulting from consecutive convolution (German: Faltung) of the frequency-function $p(z)$. The mean sum at risk of $p^*(z)$, however, is no longer equal to unity, but equal to

$$\int_0^\infty z\, p^*(z)\, dz = \frac{1}{ln(1+\chi)} \sum_{r=1}^{\infty} \left(\frac{\chi}{\chi+1}\right)^r = \frac{\chi}{ln(1+\chi)}. \qquad (18)$$

By means of the transformation (17) all the theorems and formulas known from the theory based on constant basic-probabilities may be transferred to our generalized theory, as far as they are based on the same unit-period, corresponding to the value of h_0, and as far as they refer to problems depending exclusively on the risk-business of the whole unit-periods. Most of the formulas mentioned below, which are directly proved, might also be proved by means of the transformation (17).

12) We now proceed to derive the frequency-function of the total amount x to be paid out in the special case of uniform sums at risk. Starting from formula (9) we get

$$^{(h_0)}f_1(x, P) = \int_0^\infty \frac{h_0^{h_0} e^{-h_0 q}\, q^{h_0-1}\, e^{-Pq} (Pq)^x}{\Gamma(h_0)\, x!}\, dq$$

$$= \frac{h_0^{h_0} P^x}{x!\,\Gamma(h_0)} \int_0^\infty e^{-q(h_0+P)}\, q^{x+h_0-1}\, dq$$

$$= \binom{h_0-1+x}{x} \left(\frac{P}{P+h_0}\right)^x \left(\frac{h_0}{P+h_0}\right)^{h_0}. \qquad (19)$$

The frequency-function (19) is identical with the frequency-function of Eggenberger, which is deduced from a special urn-model illustrating the frequencies of certain infectious diseases. (German: Urnenschema mit Wahrscheinlichkeitsansteckung).

Hence we conclude that the special arrangements in the urn-model of Eggenberger cause certain deviations in the basic-probabilities, which follow the main-distribution (8).

13) From the ordinary collective theory of risk we know the formula

$$F(x, P) = \sum_{r=0}^{\infty} \frac{e^{-P} P^r}{r!} \int_0^x p^{(r)}(z)\, dz\,. \tag{20}$$

In our generalized theory an analogous expression holds true, which we derive from the characteristic function (10) in the following manner:

$$\begin{aligned} {}^{(h_0)}_{x}\varphi_1(t, P) &= \left(\frac{h_0}{P+h_0}\right)^{h_0} \left[1 - \frac{P}{P+h_0}\, {}_z\pi(t)\right]^{-h_0} \\ &= \left(\frac{h_0}{P+h_0}\right)^{h_0} \left\{1 + \binom{h_0}{1} \frac{P}{P+h_0}\, {}_z\pi(t) + \binom{h_0+1}{2} \left(\frac{P}{P+h_0}\right)^2 {}_z\pi^2(t) \ldots \right\} \\ &= \sum_{r=0}^{\infty} \left[\binom{h_0-1+r}{r} \left(\frac{P}{P+h_0}\right)^r \left(\frac{h_0}{P+h_0}\right)^{h_0} {}_z\pi^r(t)\right]. \end{aligned}$$

The last formula can easily be transferred to probability-functions, and we obtain analogously to formula (20)

$$^{(h_0)}F_1(x, P) = \sum_{r=0}^{\infty} \binom{h_0-1+r}{r} \left(\frac{P}{P+h_0}\right)^r \left(\frac{h_0}{P+h_0}\right)^{h_0} \int_0^x p^{(r)}(z)\, dz, \tag{20 a}$$

which is identical with formula (20) for $h_0 = \infty$.

14) In the following we deduce the limiting probability-function for large values of P. For this purpose we introduce the auxiliary-variable

$$u' = \frac{x-P}{\sqrt{M_2'(x)}} = \frac{x-P}{P\sqrt{p_2 P^{-1} + h_0^{-1}}},$$

wherein in the denominator we have the standard deviation of x. Starting from the characteristic function (10), we obtain by means of the usual substitution-rules

$$ {}^{(h_0)}_{u'}\varphi_1(t,P) = e^{-\frac{itP}{P\sqrt{p_2P^{-1}+h_0^{-1}}}}\left\{1-\frac{P}{h_0}\left[{}_z\pi\left(\frac{t}{P\sqrt{p_2P^{-1}+h_0^{-1}}}\right)-1\right]\right\}^{-h_0} $$

$$ = e^{-\frac{it}{\sqrt{p_2P^{-1}+h_0^{-1}}}} \cdot $$

$$ \cdot\left\{1-\frac{P}{h_0}\int\limits_0^\infty p(z)dz\left(\frac{itz}{P\sqrt{p_2P^{-1}+h_0^{-1}}}+\frac{(itz)^2}{2!}\,\frac{1}{P^2(p_2P^{-1}+h_0^{-1})}\cdots\right)\right\}^{-h_0}. $$

When $h_0 \neq \infty$, we obtain in the limit $P \to \infty$

$$ \lim_{P\to\infty} {}^{(h_0)}_{u'}\varphi_1(t,P) = e^{-it\sqrt{h_0}}\left[\frac{h_0}{h_0-it\sqrt{h_0}}\right]^{h_0}. \tag{21} $$

For the auxiliary-variable $u = \frac{x}{P}$ we have analogously

$$ \lim_{P\to\infty} {}^{(h_0)}_{u}\varphi_1(t,P) = \left(\frac{h_0}{h_0-it}\right)^{h_0}. \tag{21 a} $$

The characteristic functions of the distributions of u and u' tend uniformly in any finite interval of t to their limiting functions. Hence the corresponding probability-functions tend to the probability-functions, which correspond to the characteristic functions (21) and (21 a). By means of the inversion-theorem we easily get corresponding to formula (21 a)

$$ \lim_{P\to\infty} {}^{(h_0)}F_1(u,P) = \int\limits_0^u \frac{h_0^{h_0}e^{-h_0q}q^{h_0-1}}{\Gamma(h_0)}dq. \tag{21 a'} $$

The expression on the right hand side is the incomplete Γ-function, identical with the probability-function of our main-distribution (8).

15) Under 14) we excluded the value $h_0 = \infty$. We now proceed to the limiting case, in which h_0 and P proportionally tend to infinity, i. e. P tends to infinity, whereas $\frac{P}{h_0} = \chi$ remains constant. This limiting case occurs, if an infinite sequence of unit-periods is united.

In the same way as in art. 14) we get for the characteristic function based on the auxiliary-variable u'

$$ {}^{(h_0)}_{u'}\varphi_1(t,P) = e^{-\frac{itP}{P^{1/2}\sqrt{p_2+\chi}}} \cdot $$

$$ \cdot \left\{ 1 - \chi \int_0^\infty p(z)\,dz \left(\frac{itz}{1!} \frac{1}{P^{1/2}\sqrt{p_2+\chi}} + \frac{(itz)^2}{2!} \frac{1}{P(p_2+\chi)} \cdots \right) \right\}^{-\frac{P}{\chi}} $$

$$ = \left\{ \left[1 + \frac{it\chi}{P^{1/2}\sqrt{p_2+\chi}} + \frac{(it)^2}{2!\,P} \frac{\chi^2}{p_2+\chi} \cdots \right] \cdot \right. $$

$$ \left. \cdot \left[1 - \frac{it\chi}{P^{1/2}\sqrt{p_2+\chi}} + \frac{(it)^2}{2!} \frac{\chi p_2}{P(p_2+\chi)} \cdots \right] \right\}^{-\frac{P}{\chi}}. $$

In the limit we obtain, uniformly in any finite t-interval

$$ \lim_{P\to\infty;\ \frac{P}{h_0}=\chi} {}^{(h_0)}_{u'}\varphi_1(t,P) = e^{-\frac{t^2}{2}}, \tag{22} $$

from which we conclude

$$ \lim_{P\to\infty;\ \frac{P}{h_0}=\chi} {}^{(h_0)}F_1(u',P) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u'} e^{-\frac{u'^2}{2}}\,du' = \Phi(u'). \tag{22'} $$

As in the theory based on constant basic-probabilities we arrive at the normal probability-function as the limiting function. Attention must be given to the fact, however, that the auxiliary variable u' assumes the value

$$ u' = \frac{x-P}{\sqrt{p_2 P}} $$

in the case of constant basic-probabilities, and the value

$$ u' = \frac{x-P}{\sqrt{P(p_2+\chi)}} $$

in the case of fluctuating basic-probabilities.

It is remarkable that both limiting cases treated above under 14) and 15), which lead to the same result in the ordinary theory based on constant basic-probabilities, lead to two different limiting distributions in our generalized theory, each differing from the limiting distribution in the ordinary theory.

c) *The Numerical Calculation of the Probability Function of the Gain.*

16) The well known method for the numerical calculation of the probability-function of the gain, developed by Esscher, can be transferred to our generalized assumptions.

Let

$$\bar{p}(z) = \frac{e^{kz}}{\nu_0} p(z) \tag{23}$$

be a transformed frequency-function of the sums at risk, their moments about the origin of the order r being defined by the expression

$$\nu_r = \int_0^\infty \bar{p}(z)\, z^r\, dz = \int_0^\infty \frac{e^{kz} z^r p(z)}{\nu_0}\, dz = \frac{\bar{\nu}_r}{\nu_0}.$$

For the frequency-functions $p^{(n)}(z)$, generated by consecutive convolution of $p(z)$, we further have

$$p^{(n)}(z) = \nu_0^n\, e^{-kz}\, \bar{p}^{(n)}(z).$$

Introducing the transformed frequency-functions $\bar{p}^{(n)}(z)$ in formula (20 a), we obtain

$$ {}^{(h_0)}f_1(x, P) = \sum_{n=0}^{\infty} \binom{h_0 - 1 + n}{n} \left(\frac{P}{h_0 + P}\right)^n \left(\frac{h_0}{h_0 + P}\right)^{h_0} \nu_0^n\, e^{-kx}\, \bar{p}^{(n)}(x)$$

$$= \sum_{n=0}^{\infty} \binom{h_0 - 1 + n}{n} \left(\frac{\bar{P}}{h_0 + \bar{P}}\right)^n \left(\frac{h_0}{h_0 + \bar{P}}\right)^{h_0} \cdot \left(\frac{h_0 + \bar{P}}{h_0 + P}\right)^{h_0} e^{-kx}\, \bar{p}^{(n)}(x), \tag{24}$$

wherein we have introduced

$$\bar{P} = \frac{P \nu_0}{1 - \frac{P}{h_0}(\nu_0 - 1)}.$$

Formula (24) may also be written

$$ {}^{(h_0)}f_1(x, P) = e^{-kx}\left[1-\frac{P}{h_0}(\nu_0-1)\right]^{-h_0} {}^{(h_0)}\bar{f}_1(x, \bar{P}), \quad (25) $$

in which ${}^{(h_0)}\bar{f}_1(x, \bar{P})$ denotes the transformed frequency-function, constructed in the same manner as ${}^{(h_0)}f_1(x, P)$, but P and $p(z)$ are replaced by $\bar{P}$ and $\bar{p}(z)$. Putting $uP = x$, we obtain integrating the expression (25)

$$ {}^{(h_0)}F_1(uP, P) = $$

$$ = \left[1-\frac{P}{h_0}(\nu_0-1)\right]^{-h_0} e^{-ukP}\int_0^{uP} e^{-k(z-uP)}\cdot {}^{(h_0)}\bar{f}_1(z, \bar{P})\,dz \quad (26\text{ a}) $$

$$ = 1-\left[1-\frac{P}{h_0}(\nu_0-1)\right]^{-h_0} e^{-ukP}\int_{uP}^{\infty} e^{-k(z-uP)}\cdot {}^{(h_0)}\bar{f}_1(z, \bar{P})\,dz. \quad (26\text{ b}) $$

The function

$$ \psi = \left[1-\frac{P}{h_0}(\nu_0-1)\right]^{-h_0} e^{-ukP} $$

occurring in the formulas (26), reaches its minimum-value for given values of u, if the quantity k — hitherto arbitrary — will be determined from the equation

$$ u = \frac{\bar{\nu}_1}{1-\frac{P}{h_0}(\nu_0-1)} = \frac{\int_0^{\infty} z\,e^{kz}p(z)\,dz}{1-\frac{P}{h_0}\left(\int_0^{\infty} e^{kz}p(z)\,dz-1\right)}. \quad (27) $$

In this way we always have $k \gtreqless 0$ for $u \gtreqless 1$. On further examination it may be shown, that the integrals occurring in the formulas (26) become < 1, if formula (26 a) is used, when $0 < u \leq 1$, and formula (26 b), when $u \geq 1$. Thus we have the inequalities

$$ {}^{(h_0)}F_1(uP, P) < \psi \qquad 0 < u \leq 1 \quad (28\text{ a}) $$

$$ 1-{}^{(h_0)}F_1(uP, P) < \psi. \qquad u \geq 1 \quad (28\text{ b}) $$

A closer valuation of the probability-functions we get by evaluating the integrals in the formulas (26). Normalizing first the probability-function ${}^{(h_0)}\overline{F}_1(x, \overline{P})$ by putting

$$x = \overline{P}\nu_1 + \xi\sqrt{\overline{P}\nu_2 + \overline{P}^2 h_0^{-1}} = \overline{P}\nu_1 + \xi\sqrt{\overline{M}_2'},$$

we obtain

$$^{(h_0)}\overline{F}_1(uP, P) = \psi\int_{-\infty}^{0} e^{-k\xi\sqrt{\overline{M}_2'}}\,[{}^{(h_0)}\overline{f}_1(\xi, \overline{P})]\,d\xi \qquad o < u \leq 1$$

$$1 - {}^{(h_0)}\overline{F}_1(uP, P) = \psi\int_{0}^{\infty} e^{-k\xi\sqrt{\overline{M}_2'}}\,[{}^{(h_0)}\overline{f}_1(\xi, \overline{P})]\,d\xi, \qquad u \geq 1$$

in which $[{}^{(h_0)}\overline{f}_1(\xi, \overline{P})]$ denotes the normalized frequency-function. As shown under (15) we have for $[{}^{(h_0)}\overline{F}_1(\xi, \overline{P})]$ the limiting formula

$$\lim_{\overline{P}\to\infty;\ \frac{P}{h_0}=\chi} [{}^{(h_0)}\overline{F}_1(\xi, \overline{P})] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-\frac{\xi^2}{2}}\,d\xi = \Phi(\xi). \tag{29}$$

Putting for abbreviation

$$A_0(y) = \int_{0}^{\infty} e^{\xi y}\,d\,\Phi(\xi)$$

and replacing $[{}^{(h_0)}\overline{F}_1(\xi, \overline{P})]$ by its limiting function $\Phi(\xi)$, we arrive at the formulas

$$^{(h_0)}\overline{F}_1(uP, P) \sim \psi A_0(-k\sqrt{\overline{M}_2'}) \qquad o < u \leq 1 \tag{30 a}$$

$$1 - {}^{(h_0)}\overline{F}_1(uP, P) \sim \psi A_0(k\sqrt{\overline{M}_2'}). \qquad u \geq 1 \tag{30 b}$$

Even more accurate results we obtain, if we approximate $[{}^{(h_0)}\overline{F}_1(\xi, \overline{P})]$ by means of the two first terms of the A-series

$$[{}^{(h_0)}\overline{F}_1(\xi, \overline{P})] \sim \Phi(\xi) \quad \beta_3\,\Phi'''(\xi),$$

where

$$\beta_3 = \frac{\overline{M}_3'}{6\,\overline{M}_2'^{3/2}}.$$

Putting for abbreviation

$$A_3(y) = \int_0^\infty e^{\xi y} \Phi'''(\xi)$$

we get — in comparison with the formulas (30) — the even better formulas

$${}^{(h_0)}F_1(uP,P) \sim \psi[A_0(-k\sqrt{\overline{M}_2'}) + \beta_3 A_3(-k\sqrt{\overline{M}_2'})] \quad o < u \leq 1 \quad (31\,\text{a})$$

$$1 - {}^{(h_0)}F_1(uP,P) \sim \psi[A_0(k\sqrt{\overline{M}_2'}) - \beta_3 A_3(k\sqrt{\overline{M}_2'})], \qquad u \geq 1 \quad (31\,\text{b})$$

in which the moments $\overline{M}_2$ and $\overline{M}_3$ must be calculated by means of the expressions

$$\overline{M}_2' = (uP)\frac{\bar{\nu}_2}{\bar{\nu}_1} + \frac{(uP)^2}{h_0}$$

$$\overline{M}_3' = (uP)\frac{\bar{\nu}_3}{\bar{\nu}_2} + 3(uP)^2\frac{\bar{\nu}_2}{\bar{\nu}_1 h_0} + 2\frac{(uP)^3}{h_0^2}.$$

Finally we point out, that the formulas deduced in this part may also be derived by means of the transformation (17).

d) *Numerical examples.*

17) We now proceed to study the numerical effect of the introduction of fluctuating basic-probabilities in the probability-function of the gain. Thereby we see that there is a significant difference between small and large values of P. In the case of small values of P the probability-function is nearly the same, whether we take account of fluctuating basic-probabilities or not. Therefore calculations could always be made by means of constant basic-probabilities in the case of small values of P.

Quite different, however, is the case of large values of P, as will be shown in some numerical examples. First let us consider the case of uniform sums at risk; in this simple special-case the probability-function of the gain is discontinuous for all integer values of x. For practical reasons we calculate it, however, by integrating the frequency-function (19) like a

Table 1.

x	$^{(h_0)}F_1(x, P)\,10^3$						x
	$P = 100$			$P = 1\,000$			
	$h_0=101$	$h_0=1001$	$h_0=\infty$	$h_0=101$	$h_0=1001$	$h_0=\infty$	
70	11.5	1.3	0.8	0.7	0.0	0.0	700
80	72.2	24.9	20.0	21.7	0.0	0.0	800
90	244.5	170.4	158.4	169.1	11.2	0.7	900
100	514.1	507.6	506.7	513.3	504.5	502.2	1 000
110	766.4	830.4	841.5	832.0	985.9	999.0	1 100
120	917.3	968.6	974.6	967.0	999.9	999.9	1 200
130	978.3	996.9	998.0	996.2			1 300

continuous function, whereby we integrate by means of Simpson's rule, using certain equidistant values of the frequency-function (19).

If we compare corresponding values calculated by different values of h_0, we find that the probability-function depends considerably on h_0. Therefore there is no justification for neglecting the fluctuations of the basic-probabilities, when P is large.

18) From the practical point of view, it is interesting to see, how close the probability-function may be evaluated by means of Esschers method generalized under 16). In order to examine this question, we compare the values given in table 1, calculated directly with values calculated by means of the formulas (28), (30) and (31) (table 2).

In a further example we assume $p(z) = e^{-z}$ and evaluate the probability-function by means of the formulas (28), (30) and (31). In order to compare these values we calculate the same values also using formula (20 a), whereby we evaluate the sum like an integral according to Simpson's rule, and by putting approximately

$$\int_0^x p^{(r)}(z)\,dz \sim \Phi\left(\frac{x-r}{\sqrt{r}}\right) - \frac{1}{6\sqrt{r}}\,\Phi'''\left(\frac{x-r}{\sqrt{r}}\right),$$

Table 2.

u %	$^{(h_0)}F_1(uP, P)\,10^3$ calculated by means of the formula			
	(28)	(30)	(31)	(19)
	$P = 100,\ h_0 = 101$			
80	< 322.0	68.8	71.0	72.2
100	<1 000.0	500.0	511.8	514.1
120	> 599.6	914.5	916.7	917.3
140	> 965.5	995.6	995.7	995.7
	$P = 100,\ h_0 = 1\,001$			
80	< 140.1	24.4	24.8	24.9
100	<1 000.0	500.0	508.7	507.6
120	> 816.8	968.0	968.5	968.6
140	> 998.3	999.8	999.8	999.8
	$P = 1\,000,\ h_0 = 101$			
80	< 121.8	21.1	21.5	21.7
100	<1 000.0	500.0	507.6	513.3
120	> 804.4	965.0	965.6	967.0
140	> 997.2	999.7	999.7	999.7
	$P = 1\,000,\ h_0 = 1\,001$			
80	< 0.0	0.0	0.0	0.0
100	<1 000.0	500.0	503.7	504.5
120	> 999.9	999.9	999.9	999.9
140	> 999.9	999.9	999.9	999.9

following from the expansion in the *A*-series. Of course, the values calculated by this method contain a certain error of approximation, so that the last places, in particular for small values of u, are not significant (table 3).

Comparing the corresponding values calculated by different methods, we state quite a satisfying agreement, in particular for large values of u, where even the inequality (28) may give a sufficient evaluation of the probability-function. This last mentioned quality is important for many applications, where often only large values of u must be considered.

Table 3.

u %	${}^{(h_0)}F_1(uP, P)\,10^3$ calculated by means of formula			
	(28)	(30)	(31)	(20 a)
	$P = 100,\ h_0 = 101$			
80	< 473.1	114.4	119.7	120.1
100	<1 000.0	500.0	517.9	516.6
120	> 455.1	868.3	874.0	873.7
140	> 892.5	983.8	984.4	984.8
	$P = 100,\ h_0 = 1\,001$			
80	< 345.9	75.2	78.0	79.0
100	<1 000.0	500.0	514.5	512.2
120	> 579.9	908.7	911.8	912.3
140	> 959.1	994.6	994.8	995.2

Finally we point out that the amount of work for the evaluation of the different formulas is nearly the same, if we take account of fluctuating or constant basic-probabilities.

III. Some Remarks Concerning the Ruin-Problem.

19) Besides the measures of risk based on the probability-function of the gain the probability of ruin or the adjustment-index is an important measure of risk in the collective theory of risk. This probability can also be determined under the assumption of fluctuating basic-probabilities. A detailed discussion of this subject, however, must be reserved for further investigations; only one important special case will be treated on its general lines.

Let us consider an insurance-institution, which only carries on the pure risk business. The gross-risk-premiums containing a proportional security-loading λ, form a risk-reserve SR starting from an initial value u (measured with the mean sum at risk), from which the sums at risk fallen due are paid out. Our problem consists of determining the probability that this risk-reserve SR becomes negativ a) at any moment or b) at the end of any unit-period in the future.

From the theory based on constant basic-probabilities, we know that the probabilities $\psi(u)$ [a)] and $\psi_P(u)$ [b)] satisfy the inequality

$$\psi_P(u) < \psi(u) < e^{-Ru}, \tag{32}$$

where the adjustment-coefficient R is defined by

$$\int_0^\infty e^{Rz} p(z)\, dz = 1 + (1+\lambda) R. \tag{33}$$

For large values of u we further have the asymptotic expressions

$$\lim_{u\to\infty} \frac{\psi(u)}{e^{-Ru}} = \frac{\lambda}{\int_0^\infty z e^{Rz} p(z)\, dz - 1 - \lambda} \tag{34 a}$$

$$\lim_{u\to\infty} \frac{\psi_P(u)}{e^{-Ru}} = \frac{1}{1+\lambda R P} \, \frac{\lambda}{\int_0^\infty z e^{Rz} p(z)\, dz - 1 - \lambda}, \tag{34 b}$$

wherein P denotes as usual the net-risk-premium, which corresponds to a unit-period.

20) By means of the transformation (17) we can transfer the probability (34 b), valid for constant basic-probabilities, to the case of fluctuating basic-probabilities, whereby special attention must be given to the fact that the mean sum at risk of $p^*(z)$ is not equal to unity, but given by the expression (18).

In order to derive the adjustment coefficient R we start from the relation

$$\int_0^\infty e^{Rz} p^*(z)\, dz = 1 + (1+\lambda) R \int_0^\infty z p^*(z)\, dz,$$

and obtain, assuming that the integral on the left hand side converges, after some reduction

$$\int_0^\infty e^{Rz} p(z)\,dz = 1 + \frac{1 - e^{-(1+\lambda)R\chi}}{\chi} \tag{35}$$

in the place of formula (33). Formula (35) shows that the convergence assumed indeed holds true, because

$$\int_0^\infty e^{Rz} p(z)\,dz < \frac{\chi + 1}{\chi},$$

so that the series, which we get by means of formula (17), converges.

In an analogous manner we transfer formula (34 b) to our generalized assumptions; thereby we must in particular pay attention to the relation

$$\int_0^\infty z\, e^{Rz} p^*(z)\,dz = \frac{d}{dR}\int_0^\infty e^{Rz} p^*(z)\,dz;$$

after some reduction we get

$$\lim_{u\to\infty} \frac{\psi_P(u)}{e^{-Ru}} = \frac{\lambda}{e^{(1+\lambda)R\chi}\int_0^\infty z\,e^{Rz}p(z)\,dz - 1 - \lambda}\;\frac{1}{1+\lambda R P} \tag{36}$$

or approximately

$$\psi_P(u) \sim e^{-Ru}. \tag{37}$$

In table 4 we give some examples of the probability $\psi_P(u)$, where we assume $p(z) = e^{-z}$, $u = P = 1\,000$ and $h_0 = 100$. In

Table 4.

λ %	$\psi_{1000}(1\,000)$ calculated by means of the formulas (33)/(34 b)	(35)/(36)
10	0.3 10^{-40}	0.2 10^{-8}
20	0.1 10^{-73}	0.3 10^{-11}
30	0.7 10^{-102}	0.7 10^{-17}
40	0.5 10^{-126}	0.2 10^{-20}

order to show the numerical effect of fluctuating basic-probabilities, we calculate $\psi_P(u)$ by means of the formulas (35)/(36) as well as using the formulas (33)/(34 b).

Table 4 shows that the order of magnitude of the probability $\psi_P(u)$ changes in a fundamental manner by introducing fluctuating basic-probabilities. Therefore it will not do to neglect the fluctuations of the basic-probabilities, not even as a first approximation.

Final Remarks.

In this paper we have tried to generalize the collective theory of risk by introducing the assumption of fluctuating basic-probabilities. Under such assumptions the results obtained are often as regards the insurance field, in better accordance with the real circumstances than the results of the usual theory. Our investigations described above show, that this generalization does not lead to considerable difficulties, neither from the theoretical nor from the practical point of view. Thus it may be possible to put into practice the theory of risk to a much larger extent than it has been the case hitherto.

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