Moment explosion in the LIBOR market model

Stefan Gerhold

Vienna University of Technology, Institute of Mathematical Methods in Economics, Wiedner Hauptstr. 8/105-1, A-1040 Vienna, Austria

ARTICLE INFO

Article history:
Received 18 August 2010
Received in revised form 15 December 2010
Accepted 14 January 2011
Available online 22 January 2011

MSC: 91G30

Keywords:
LIBOR market model
Moment explosion
Distribution tails

ABSTRACT

In the LIBOR market model, forward interest rates are log-normal under their respective forward measures. This note shows that their distributions under the other forward measures of the tenor structure have approximately log-normal tails.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

The LIBOR market model (Brace et al., 1997) is one of the most popular models for pricing and hedging interest rate derivatives. Its state variables are forward interest rates $F_n(t) := F(t; T_{n-1}, T_n)$, spanning time periods $[T_{n-1}, T_n]$, where

$0 < T_0 < T_1 < \cdots < T_M$

is a fixed tenor structure. Under the $T_M$-forward measure $\mathbb{Q}^M$, which has as numeraire the zero-coupon bond maturing at $T_M$, the dynamics of the forward rates are

$$
\frac{dF_n(t)}{F_n(t)} = \rho_{n\ell} \tau F_j(t) \, dt + \sigma_n(t) dW_n(t), \quad 1 \leq n < M,
$$

$$
\frac{dF_M(t)}{F_M(t)} = \sigma_M(t) dW_M(t).
$$

Here, the $\sigma_n$ are some positive deterministic volatility functions, and $W$ is a vector of standard Brownian motions with instantaneous correlations $dW_i(t) dW_j(t) = \rho_{ij} dt$. Moreover, $\tau_n = \tau(T_{n-1}, T_n)$ denotes the year fraction between the tenor dates $T_{n-1}$ and $T_n$.

Note that $F_M$ is a geometric Brownian motion under $\mathbb{Q}^M$. More generally, each rate $F_n$ is a geometric Brownian motion under its own forward measure, while it has a stochastic drift under the other forward measures. Therefore, pricing derivatives with the LIBOR market model often requires a Monte Carlo simulation. To avoid this, a popular approximation of the above dynamics is obtained by “freezing the drift” (Brace et al., 2001; Brigo and Mercurio, 2006):

$$
\frac{dF_{n}^{ld}(t)}{F_{n}^{ld}(t)} = \rho_{n\ell} \tau F_j(0) \, dt + \sigma_n(t) F_{n}^{ld}(t) dW_n(t), \quad 1 \leq n < M,
$$

$$
\frac{dF_{M}^{ld}(t)}{F_{M}^{ld}(t)} = \sigma_M(t) F_{M}^{ld}(t) dW_M(t).
$$

E-mail address: sgerhold@fam.tuwien.ac.at

0167-7152/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2011.01.009
Since the log-drifts are now deterministic, the new rates \( F_{n}^{fd} \) are just geometric Brownian motions, which allow for explicit pricing formulas for many interest-linked products. As a piece of evidence for the quality of this approximation, we show in the present note that, for fixed \( t > 0 \), the distribution of \( F_{n}^{fd}(t) \) has roughly the same tail heaviness as the distribution of \( F_{n}(t) \).

2. Main result

If \( X \) is any log-normal random variable, and so \( \log X \sim N(\mu, \sigma^{2}) \) for some real \( \mu \) and positive \( \sigma \), then

\[
\sup \{ v : E[e^{v \log^{2} X}] < \infty \} = \frac{1}{2\sigma^{2}}.
\]

This follows from

\[
E[e^{v \log^{2} X}] = \frac{1}{\sqrt{1 - 2\sigma^{2}v}} \exp \left( \frac{\mu^{2}v}{1 - 2\sigma^{2}v} \right), \quad v < \frac{1}{2\sigma^{2}}.
\]

Our main result shows that \( F_{n}(t) \) has approximately log-normal tails, in the sense that the left-hand side of (1) is finite and positive if \( X \) is replaced by \( F_{n}(t) \). Furthermore, this “critical moment” is the same for \( F_{n}(t) \) and the frozen drift approximation \( F_{n}^{fd}(t) \). We write \( E^{n} \) for the expectation w.r.t. the \( T_{n} \)-forward measure \( \mathbb{Q}^{n} \).

**Theorem.** In the log-normal LIBOR market model, we have for all \( t > 0 \) and all \( 1 \leq n \leq M \)

\[
\sup \{ v : E^{n}[e^{v \log^{2} F_{n}(t)}] < \infty \} = \frac{1}{2} \int_{0}^{t} \sigma_{n}(s)^{2}ds.
\]

**Proof.** Note that the latter equality is obvious from (1), since \( F_{n}^{fd}(t) \) is log-normal with log-variance parameter \( \sigma^{2} = \int_{0}^{t} \sigma_{n}(s)^{2}ds \). We now show the first equality. Recall that the measure change from the \( T_{n} \)-forward measure to the \( T_{n-1} \)-forward measure is effected by the likelihood process (Björk, 2004)

\[
\frac{d\mathbb{Q}^{n}}{d\mathbb{Q}^{n-1}} |_{\mathcal{F}_{t}} = \frac{1 + \tau_{n} F_{n}(0)}{1 + \tau_{n} F_{n}(t)}.
\]

Therefore, putting \( \phi(x) = \exp(\log^{2} x) \), we obtain

\[
E^{n}[\phi(F_{n}(t))^{v}] = E^{n-1} \left[ \phi(F_{n}(t))^{v} \times \frac{1 + \tau_{M} F_{M}(0)}{1 + \tau_{M} F_{M}(t)} \right] = \cdots = E^{n} \left[ \phi(F_{n}(t))^{v} \prod_{i=n+1}^{M} \frac{1 + \tau_{i} F_{i}(0)}{1 + \tau_{i} F_{i}(t)} \right] \leq E^{n}[\phi(F_{n}(t))^{v}] \prod_{i=n+1}^{M} (1 + \tau_{i} F_{i}(0)),
\]

and hence

\[
\sup \{ v : E^{n}[\phi(F_{n}(t))^{v}] < \infty \} \leq \sup \{ v : E^{M}[\phi(F_{n}(t))^{v}] < \infty \}.
\]

On the other hand, for \( 1 < k < M \) and \( v \in \mathbb{R} \) we have

\[
E^{k-1}[\phi(F_{n}(t))^{v}] = E^{k} \left[ \phi(F_{n}(t))^{v} \times \frac{1 + \tau_{k} F_{k}(t)}{1 + \tau_{k} F_{k}(0)} \right] = \frac{1}{1 + \tau_{k} F_{k}(0)} (E^{k}[\phi(F_{n}(t))^{v}] + \tau_{k} E^{k}[F_{k}(t)\phi(F_{n}(t))^{v}]).
\]

Now let \( \varepsilon > 0 \) be arbitrary, and define \( q \) by \( \frac{1}{q} + \frac{1}{1+\varepsilon} = 1 \). Then Hölder’s inequality yields

\[
E^{k}[F_{k}(t)\phi(F_{n}(t))^{v}] \leq E^{k}[F_{k}(t)^{\varepsilon}]^{1/q} \times E^{k}[\phi(F_{n}(t))^{v(1+\varepsilon)}]^{1/(1+\varepsilon)}.
\]

By the finite moment assumption, we obtain the implication

\[
E^{k}[\phi(F_{n}(t))^{v(1+\varepsilon)}] < \infty \implies E^{k-1}[\phi(F_{n}(t))^{v}] < \infty, \quad v \in \mathbb{R}.
\]

(Note that the left-hand side implies \( E^{k}[\phi(F_{n}(t))^{v}] < \infty \).)
Inductively, this leads to the implication
\[ E^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty \implies E^M[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}. \]

Therefore, we find
\[
sup\{v : E^M[\phi(F_n(t))^v] < \infty\} \geq sup\{v : E^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty\} = \frac{1}{(1+\varepsilon)^{M-n}} sup\{v : E^M[\phi(F_n(t))^v] < \infty\}.
\]

Since \( \varepsilon \) was arbitrary,
\[
sup\{v : E^M[\phi(F_n(t))^v] < \infty\} \geq sup\{v : E^M[\phi(F_n(t))^v] < \infty\}
\]
follows, which finishes the proof. \( \square \)

The following corollary explicitly shows the implication of Theorem 1 on the distribution tails of \( F_n(t) \): At least one of the two tails is log-normal, and neither tail is heavier than log-normal. We are talking here about the dominating factor of the tail asymptotics; a lower order factor (a power of \( x \), for example) could make the tails (slightly) heavier or lighter than log-normal. These refined asymptotics are left for future research. Note also that our result does not so far exclude the possibility that one of the two tails is (significantly) lighter than log-normal.

**Corollary 2.** Under the assumptions of Theorem 1, the supremum of all real \( v \) that satisfy the condition
\[
Q^M[F_n(t) > x] + Q^M[F_n(t) < \frac{1}{x}] = O(e^{-v \log^2 x}), \quad x \to \infty,
\]
is given by
\[
v^* = \frac{1}{2 \int_0^1 \sigma_n(s)^2 \, ds}.
\]

The same holds for \( F^{fd}_n(t) \) instead of \( F_n(t) \).

**Proof.** From Theorem 1 we obtain
\[
v^* = sup\{v : Q^M[e^{\log^2 F_n(t)} > x] = O(x^{-v})\}.
\]

Now for \( x > 1 \) we have
\[
Q^M[e^{\log^2 F_n(t)} > x] = Q^M[F_n(t) > e^{\log x}] + Q^M[F_n(t) < e^{-\log x}],
\]
and the claim follows after replacing \( x \) by \( \exp(\log^2 x) \). The same reasoning works for the frozen drift rate \( F^{fd}_n(t) \). \( \square \)

**References**


