How to make Dupire's local volatility work with jumps

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How to make Dupire's local volatility work with jumps¶

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Dedicated to the memory of Marc Yor

There are several (mathematical) reasons why Dupire's formula fails in the non-diffusion setting. And yet, in practice, ad-hoc preconditioning of the option data works reasonably well. In this note, we attempt to explain why. In particular, we propose a regularization procedure of the option data so that Dupire's local vol diffusion process recreates the correct option prices, even in manifest presence of jumps.

1. Failure of Dupire's formula in non-diffusion setting

Local volatility models (Dupire 1994, Dupire 1996), '$dS/S = \sigma_{\mathrm{loc}}(S, t)dW$', are a must for every equity option trading floor. A central ingredient is *Dupire's formula*, which allows obtaining the diffusion coefficient σ_{loc} directly from the market (or a more complicated reference model),

$$\sigma_{\mathrm{loc}}^2(K, T) = 2\partial_T C / K^2 \partial_{KK} C, \tag{1}$$

in terms of (call) option prices at various strikes and maturities.‖ Of course, there are ill-posedness issues concerning how to compute derivatives when only given discrete (market) data; this inverse problem is usually solved by fitting market (option or implied vol) data via a smooth parametrization, from which σ_{loc} is then computed.

On a more fundamental level, given an arbitrage-free option price surface

$$\{C(K, T) : K \geq 0, T \geq 0\},$$

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¶The last author, Marc Yor, passed away shortly after final acceptance of this manuscript.
‖Throughout, we assume a zero interest rate for simplicity. Extension to a constant rate $r > 0$ is straightforward; see Remark 2.

there are two problems in a non-diffusion setting:

- *Lack of smoothness* ($\partial_{KK}C$ is precisely the stock price density). If the underlying S_T admits a continuous density, then call prices are always C^2 w.r.t. strike (Musiela and Rutkowski 2005, Cor. 7.3.1). But consider, for instance, the asymmetric Variance Gamma (AVG; cf. Cont and Voltchkova (2005)) process (X_t) with characteristic function

$$\mathbb{E}[\exp(iuX_T)] = \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2) u^2}\right)^{T/\nu}$$
$$\sim const \cdot u^{-2T/\nu} \quad \text{as} \quad u \to \infty.$$

When $2T/\nu > 1$ (equivalently: $T > \nu/2$), the characteristic function is integrable on $\mathbb{R}$ and X_T admits a continuous density; the same is then true for $S_T = S_0 \exp(X_T + \mu T)$. This is not the case, however, for $T < \nu/2$, and indeed the density (given explicitly e.g. in Madan *et al.* (1998, p. 83)) has a singularity at the origin, so that C cannot possibly be twice continuously differentiable in K.

- *Blowup of the short end of the local volatility surface.* Assume, for the sake of argument, that

$$S_T = S_0 \exp(X_T + \sigma W_T + \mu T),$$

where X is a pure jump Lévy process with Lévy measure $L = L(dx)$, and W is a standard Brownian motion. It is known (see Cont and Gu (2012)) that the local volatility of jump diffusion models explodes for small maturities. For instance, in the Merton model, where X_T has normally distributed jumps, one can see

$$\sigma_{\text{loc}}^2(K, T) \approx const \cdot \frac{1}{T} \quad \text{as } T \to 0, \ K \neq S_0. \tag{2}$$

In fact, this blowup can be fully quantified, thanks to a recently established saddle point formula (De Marco *et al.* 2013); details are given in the appendix. There we show that the approximation (2) is also true in the NIG (normal inverse Gaussian) model, whereas in Kou's double exponential jump diffusion model the blowup is of order $T^{-1/2}$. (The analysis is not restricted to exponential Lévy models. Indeed, general jump diffusion resp. semimartingales are 'tangent', at a given point in space-time, to a Lévy process, cf. Jacod and Shiryaev (2003), Stroock (1975).)

At least from a mathematical point of view, all this is strong evidence that (1) is not meaningful in presence of jumps. In fact, if one views (1) as a (forward) PDE for call option prices as function of K, T, then the analogous formula in a jump setting is a (forward) PIDE, which is thus the 'natural generalization' of Dupire's formula in presence of jumps; cf. Bentata and Cont (2010) and the references therein, in particular towards so-called local Lévy models (Carr *et al.* 2004). Note that the calibration of a local Lévy model is a numerically challenging inverse problem (Kindermann and Mayer 2011).

For good or bad, practitioners use (1) no matter what. Smoothness is usually not an issue since local vol is typically obtained from a smooth parametrization of market (implied vol) data.† On the other hand, the short-time blowup of local volatility, $\sigma_{\text{loc}}(\cdot, T) \to \infty$ as $T \to 0$, is an immediate obstacle, for it already makes it unclear to what extent there exists a (unique) strong solution to the stochastic differential equation $dS/S = \sigma_{\text{loc}}(S, t)\, dW$, let alone how to sample from it. It is reported from practitioners that various ad-hoc truncation and mollification procedures are in place, say with parameter ε, after which the local volatility is not explosive anymore; Monte Carlo simulations of '$dS^\varepsilon/S^\varepsilon = \sigma_{\text{loc}}^\varepsilon(S^\varepsilon, t)dW$' are then possible. There is every practical evidence that

$$\mathbb{E}[(S_T^\varepsilon - K)^+] \approx C(K, T)$$

is a good approximation (for otherwise, risk management would not allow this in practice). And yet, to our knowledge, there has been no mathematical justification to date for this type of approximation.

2. Regularization of call prices: how to make Dupire work

We now consider the situation of a given martingale (S_t) which creates a smooth call price surface, with (strict) absence of calendar and butterfly spreads. This situation is typical in the industry. (Of course, only option data are known, but upon

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†Doing so without introducing arbitrage is decisively non-trivial, see Gatheral and Jacquier (2011), Gatheral and Jacquier (2012).

a suitable parametrization thereof, one is in the just described situation.) In particular, $\sigma_{\text{loc}}^2(K, T) = 2\partial_T C / K^2 \partial_{KK} C$ is well-defined, as long as $T > 0$, but may explode as $T \to 0$ (thereby indicating the possibility of jumps). This situation is also typical in a generic non-degenerate jump diffusion setting.‡

THEOREM 1 *Assume that (S_t) is a martingale (possibly with jumps) with associated call price surface C,*

$$\forall K, T \geq 0: \quad C(K, T) = \mathbb{E}[(S_T - K)^+],$$

smooth (i.e. C^∞) in $\{(K, T) : K > 0, T > 0\}$, and such that $\partial_T C > 0$ and $\partial_{KK} C > 0$, i.e. (strict) absence of calendar and butterfly spreads. Define ε-shifted local volatility

$$\sigma_\varepsilon^2(K, T) = \frac{2\partial_T C(K, T + \varepsilon)}{K^2 \partial_{KK} C(K, T + \varepsilon)}. \tag{3}$$

Then $dS^\varepsilon/S^\varepsilon = \sigma_\varepsilon(S^\varepsilon, t)dW$, started at randomized spot S_0^ε with distribution

$$\mathbb{P}[S_0^\varepsilon \in dK]/dK = \partial_{KK} C(K, \varepsilon), \tag{4}$$

admits a unique, non-explosive strong SDE solution such that

$$\forall K, T \geq 0: \quad \mathbb{E}[(S_T^\varepsilon - K)^+] \to C(K, T) \quad \text{as } \varepsilon \to 0. \tag{5}$$

Our assumptions encode that the model itself, i.e. the specification of the dynamics of S, has regularization effects built in. The result is then, in essence, a variation of the arguments put forward in a recent revisit of Kellerer's theorem, see Hirsch *et al.* (2012) and the references therein.

Proof By assumption, σ_ε^2 is well-defined for all $T \geq 0$. (In general, i.e. without adding ε and in presence of jumps, local vol is not well-defined in the sense that it may blow up as $T \to 0$.) Existence of a unique non-explosive local vol SDE solution for continuous and locally bounded diffusion coefficient is a classical result of the theory of (one-dimensional) SDEs. Set $a^\varepsilon(S, t) = \frac{1}{2} S^2 \sigma_\varepsilon^2(S, t)$; the generator of S^ε reads $L = L_t = a^\varepsilon \partial_{SS}$. Set also $C^\varepsilon(K, T) = C(K, T + \varepsilon)$ and $p^\varepsilon = \partial_{KK} C^\varepsilon$, the density of $S_{T+\varepsilon}$. By definition,

$$a^\varepsilon(K, T) = \frac{\partial_T C^\varepsilon(K, T)}{p^\varepsilon(K, T)},$$

and hence, using $\partial_{TKK} C^\varepsilon(K, T) = \partial_T p^\varepsilon$,

$$\partial_{KK}\left(a^\varepsilon p^\varepsilon\right) = \partial_T p^\varepsilon. \tag{6}$$

In particular, p^ε (which is $L^1(0, \infty) \cap C^\infty(0, \infty)$ in K) is a (classical) solution to the above Fokker–Planck equation. On the other hand, S^ε solves the martingale problem for L_t in the sense that for any test function φ, say, C^∞-smooth with compact support,

$$t \mapsto \varphi\left(S_t^\varepsilon\right) - \varphi\left(S_0^\varepsilon\right) - \int_0^t L_s \varphi\left(S_s^\varepsilon\right) ds$$

is a zero-mean martingale. Taking expectations and writing $q^\varepsilon = q^\varepsilon(dS, t)$ for the law of S_t^ε, noting $q^\varepsilon(dS, 0) = p^\varepsilon(S, 0)dS$, we see

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‡The mathematics here is well understood (see Bichteler *et al.* (1987), Norris (1986), Norris (1988)); a smooth density (and then call prices) are typically the result of a (hypo)elliptic diffusion part, possibly aided by infinite activity jumps.

$$\int \varphi(S)q^\varepsilon(dS, t) = \int \varphi(S)q^\varepsilon(dS, 0)$$
$$+ \int_0^t \int a^\varepsilon(S, s)\varphi''(S)q^\varepsilon(dS, s),$$

which is nothing but an (analytically weak, in space) formulation of the Fokker–Planck equation (6), with solution given in terms of a family of measures, $q^\varepsilon = \{q^\varepsilon(dS, t) : t \geq 0\}$. By a suitable uniqueness result for such equations due to M. Pierre (see Hirsch *et al.* (2011)), we see that $p^\varepsilon dS = q^\varepsilon$. And we have the following consequence for call prices based on ε-regularized local vol:

$$\mathbb{E}[(S_T^\varepsilon - K)^+] = \int (S - K)^+ q^\varepsilon(dS, T)$$
$$= \int (S - K)^+ p^\varepsilon(S, T)dS$$
$$= \mathbb{E}[(S_{T+\varepsilon} - K)^+]$$
$$= C(K, T + \varepsilon).$$

Since, by assumption, C is continuous, convergence as $\varepsilon \to 0$ to $C(K, T)$ is trivial. □

Remark 2 For nonzero interest rate $r > 0$, clearly, we have to replace (3) by

$$\sigma_\varepsilon^2(K, T) = \frac{2\partial_T C(K, T + \varepsilon) + rK\partial_K C(K, T + \varepsilon)}{K^2\partial_{KK}C(K, T + \varepsilon)}.$$

The equality in distribution $S_{T+\epsilon} = S_T^\varepsilon$ then persists, and $e^{-rT}\mathbb{E}[(S_T^\varepsilon - K)^+] = C(K, T + \varepsilon)$ converges to $C(K, T)$ as $\varepsilon \to 0$.

Remark 3 Instead of $\tau_\varepsilon : T \mapsto T + \varepsilon$ one may take any continuously differentiable strictly increasing $\tau_\varepsilon : [0, \infty) \to [\varepsilon, \infty)$; in general, by the chain rule from calculus, an additional factor τ_ε' will appear in Dupire's formula.

Initial spot S_0 does not appear as input parameter; it is implicit in any marginal of S by the martingale property. It does enter a numerical implementation, as the constraints $\mathbb{E}[S_t] = S_0$ will be used when fitting a local vol surface to finitely many call prices.

The following variant of the approach presented in Theorem 1 is worth mentioning: if we do not shift T by ε in (3), we get a process $\tilde{S}^\varepsilon$ that is defined only for $t \geq \varepsilon$, but agrees with S, without a time shift: $\tilde{S}_t^\varepsilon = S_t$ in distribution, for all $t \geq \varepsilon$. Calls can thus be priced only for maturities $T \geq \varepsilon$, but the original call prices are recovered exactly, not just in the limit $\varepsilon \to 0$. The proof is a straightforward modification of the proof of Theorem 1.

THEOREM 4 *In the setting of Theorem 1, define a process* $(\tilde{S}_t^\varepsilon)_{t \geq \varepsilon}$ *by*

$$\mathbb{P}[\tilde{S}_\varepsilon^\varepsilon \in dK]/dK = \partial_{KK}C(K, \varepsilon) \qquad (7)$$

and

$$d\tilde{S}_t^\varepsilon/\tilde{S}_t^\varepsilon = \sigma_{\text{loc}}(\tilde{S}_t^\varepsilon, t)dW_t, \quad t \geq \varepsilon,$$

where $\sigma_{\text{loc}}^2(K, T) = 2\partial_T C/K^2\partial_{KK}C$. *Then, for* $K \geq 0$ *and* $T \geq \varepsilon$,

$$\mathbb{E}[(\tilde{S}_T^\varepsilon - K)^+] = C(K, T).$$

Finally, note that the stochastic initial values (4) resp. (7) are essential ingredients in the proofs. It is a natural question, whether the conclusion (5) remains valid if the deterministic value $S_0^\varepsilon = S_0$ is used instead of (4).

3. Appendix: Local volatility blowup in some jump models

Throughout this section, we normalize spot w.l.o.g. to $S_0 = 1$. At the money, i.e. for $K = S_0 = 1$, no blowup of the local volatility is to be expected. For instance, in Lévy jump diffusion models, σ_{loc} tends to the volatility σ of the jump diffusion part as $T \to 0$. To see this, recall the forward PIDE for the call price (Carr *et al.* 2004, Cont and Voltchkova 2005, Bentata and Cont 2010):

$$\partial_T C = \frac{1}{2}K^2\sigma^2\partial_{KK}C$$
$$+ \int_{-\infty}^\infty \nu(dz)\left(C(Ke^{-z}, T) - C(K, T) - K(e^z - 1)\partial_K C\right),$$

where ν is the Lévy measure. As $T \to 0$, the integral tends to a non-negative constant. The claim thus follows from (1), since the density $\partial_{KK}C$ tends to infinity for $K = S_0$.

By similar reasoning, we can quantify the off-the-money blowup of local vol, as soon as small-time asymptotics for the density in the denominator of Dupire's formula (1) are available. For example, the density of the NIG (normal inverse Gaussian) model is $\sim const \cdot T$ (in fact, there is an explicit expression for the density (Cont and Tankov 2004)). This implies $\sigma_{\text{loc}}^2(K, T) \approx const/T$ for $K \neq S_0$. (Strictly speaking, the argument gives only a *bound* $O(1/T)$. To show that the numerator $\partial_T C$ tends to a *nonzero* constant seems difficult.)

We now obtain further asymptotic results by using the local volatility approximation

$$\sigma_{\text{loc}}^2(K, T) \approx \left.\frac{2\frac{\partial}{\partial T}m(s, T)}{s(s-1)}\right|_{s=\hat{s}(K, T)} \qquad (8)$$

presented in De Marco *et al.* (2013), Friz and Gerhold (2011). Here, $m(s, T) = \log M(s, T)$ denotes the log of the moment generating function $M(s, T) = E[\exp(sX_T)]$ of the log-price $X_T = S_T$, and $\hat{s} = \hat{s}(K, T)$ solves the saddle point equation

$$\frac{\partial}{\partial s}m(s, T) = k := \log K.$$

While our focus in De Marco *et al.* (2013) was on asymptotics in K, (8) may as well be used for time asymptotics, provided that the underlying saddle point approximation can be justified. The latter can be achieved by analysing the Fourier representation of local vol, which is at the base of (8):

$$\sigma_{\text{loc}}^2(K, T) = \frac{2\partial_T C}{K^2\partial_{KK}C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)}e^{-ks}M(s, T)ds}{\int_{-i\infty}^{i\infty} e^{-ks}M(s, T)ds}. \qquad (9)$$

For instance, in Friz and Gerhold (2011) we discussed the strike asymptotics for Kou's double exponential jump diffusion, and small-time asymptotics can be done in the same way, the result being that local variance is of order $T^{-1/2}$. Note that tail integral estimates are required to give a rigorous proof; see Friz and Gerhold (2011), Friz *et al.* (2011) for two examples of such arguments. For the Kou model, the moment generating function

$$M(s, T) = \exp\left(T\left(\frac{\sigma^2 s^2}{2} + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - s} + \frac{\lambda_-(1-p)}{\lambda_- + s} - 1\right)\right)\right)$$

has a singularity of the type 'exponential of a pole', and one can copy almost verbatim the saddle point analysis of Gerhold (2011). On the other hand, the NIG model, which we discussed above, is *not* in the scope of (8). The reason is simply that there is no saddle point, as the moment generating function

$$M(s, T) = \exp\left(\delta T\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + s)^2}\right)\right)$$

has no blowup at its singularity $\alpha - \beta$.

The Merton model

The Merton jump diffusion is another case in point for the saddle point approximation (8). The resulting off-the-money blowup is essentially of order $1/T$:

$$\sigma_{\mathrm{loc}}^2(K, T) \sim const \cdot \frac{|k|}{T}\left(\log\frac{|k|}{T}\right)^{-3/2}, \quad T \to 0, \quad (10)$$

where $k = \log K \neq 0$ and the constant depends on the sign of k. To see this, recall that the Merton jump diffusion has moment generating function

$$M(s, T) = \exp\left(T\left(\frac{1}{2}\sigma^2 s^2 + bs + \lambda(e^{\delta^2 s^2/2 + \mu s} - 1)\right)\right)$$

with diffusion volatility σ, jump intensity λ, mean jump size μ and jump size variance δ^2. The saddle point $\hat{s}$ solves $\partial_s m(s, 1) = k/T$. As observed in Friz and Gerhold (2011), for Lévy models the saddle point (if it exists) is always a function of k/T, so that various asymptotic regimes can be captured by the same formula. For $k > 0$, the saddle point satisfies

$$\frac{\delta^2 \hat{s}^2}{2} = \log\frac{k}{T} - \frac{\sqrt{2}\mu}{\delta}\sqrt{\log\frac{k}{T}} - \frac{1}{2}\log\log\frac{k}{T} + \frac{\mu^2}{\delta^2}$$
$$- \log\frac{\sqrt{2}}{\delta} - \log(\lambda T\delta^2) + O\left(\frac{\log\log\frac{k}{T}}{\sqrt{\log\frac{k}{T}}}\right)$$

as $k/T \to \infty$. Inserting this into (8) yields (10), with a similar reasoning for $k < 0$. As regards the tail integral estimates needed to make this rigorous, no unpleasant surprises are to be expected, as double exponential singularities (here: at infinity) are well known to be amenable to the saddle point method.

Merton's jump-to-ruin model

In this model, the underlying (S_t) follows Black-Scholes dynamics, but may jump to zero (and stay there) at an independent exponential time with parameter λ. In other words, λ is the risk-neutral arrival rate of default. Out of the money, the additional default feature has little influence on local vol for small T, as it does not matter much that the underlying can jump even further out of the money. So we expect $\sigma_{\mathrm{loc}}^2(K, T) \to \sigma^2$ as $T \to 0$ for $K > S_0 = 1$. Indeed, this follows immediately from the explicit formula (see Gatheral (2006), Laurence (2009))

$$\sigma_{\mathrm{loc}}^2(K, T) = \sigma^2 + 2\lambda\sigma\sqrt{T}\frac{N(d_2)}{N'(d_2)}, \quad (11)$$

where N is the standard Gaussian cdf and

$$d_2 = \frac{\log(S_0/K + \lambda T)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

Formula (11) is an easy consequence of the fact that the call price in this model is just the Black-Scholes price with interest rate λ ($\lambda + r$ in general, but recall that we assume $r = 0$ throughout). The out-of-the-money convergence $\sigma_{\mathrm{loc}}^2(K, T) \to \sigma^2$ can also be confirmed by our formula (8): The moment generating function equals

$$M(s, T) = \begin{cases} \exp\left(T\left(\frac{1}{2}\sigma^2 s^2 + (\lambda - \frac{1}{2}\sigma^2)s - \lambda\right)\right) & s \geq 0 \\ \infty & s < 0. \end{cases}$$

There is a saddle point $\hat{s} = k/(\sigma^2 T) + \frac{1}{2} - \lambda/\sigma^2$, and hence (8) gives $\sigma_{\mathrm{loc}}^2(K, T) = \sigma^2 + O(T)$.

For $K < S_0 = 1$, on the other hand, (11) reveals a fast blowup of the order $\sigma_{\mathrm{loc}}^2(K, T) \approx e^{1/T}$ as $T \to 0$. Formula (8) seems not to be useful here, as the moment generating function is not defined for $\mathrm{Re}(s) < 0$, since S_T may assume the value zero.

Acknowledgements

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