THE END-OF-THE-YEAR BONUS:
HOW TO OPTIMALLY REWARD A TRADER?

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Traders are compensated by bonuses, in addition to their basic salary. However, little is
known about how to optimally reward a trader. In this article we build a framework for
the study of this problem and explore a variety of possible compensation structures.

Keywords: HJB equation; Sharpe ratio; Monte-Carlo simulations.

1. Introduction

Traders are compensated by bonuses, in addition to their basic salary. The most
simple compensation structure is to have no structure at all: the bank waits until
the end of the year, and then gives large bonuses to the traders it wants to keep. But
a bank which adopts such a policy would have no control on the traders’ trading
strategies. On the other hand, if a bonus structure is designed in advance, the bank
could assume that the trader would follow the trading strategy which maximizes
his expected bonus. The bank would therefore have some control on the trader’s trading behavior.

Of course, the story is not quite as simple as this. Traders have limits imposed on their available capital and on the gearing and type of instruments they are allowed to trade. Nevertheless little is known about how to optimally reward a trader and there is a need to develop quantitative methods to understand the consequences of a bonus structure.

In this article we are going to build a framework for the study of this problem and explore a variety of possible compensation structures.

We assume that the trader is allowed to trade only one risky asset; the price of the risky asset $S$ follows the lognormal random walk

$$dS = \mu S dt + \sigma S dX$$

where $\mu$ is the drift of the asset, $\sigma$ its volatility, and $dX$ is a Gaussian random variable with mean zero and variance $dt$.

The trader can put his money in a risk-free asset as well; his trading account $\pi$ is zero at $t = 0$, and it satisfies the following stochastic differential equation

$$d\pi = r \pi dt + q(dS - rS dt),$$

where $r$ is the risk-free interest rate and $q$ is the position of the trader, i.e. the number of risky assets that he holds. He will have some restriction such as $|q| \leq C$, $C$ is the position limit.

We introduce the function $V$, the expected bonus of the trader. It depends on $t, S, \pi$ and eventually other variables if the bonus depends on other variables. We assume that the trader will choose the trading strategy which maximizes the expected bonus: the choice of optimal strategy becomes a stochastic control problem whose solution, a second-order, non-linear, partial differential equation (PDE) in two (or more) state variables, characterises both the optimal strategy and the expected bonus. In most cases we can reduce the dimension of the problem. Once we have found the optimal trading strategy, we perform Monte-Carlo simulations of optimal trading and deduce the histograms of the trader’s profit and of the bank’s profit. For a given bonus structure the bank can therefore quantify what profit both the trader and the bank are going to make. It can therefore choose the bonus structure which fits best with its risk preferences.

In the first part of the article, we concentrate on different fixtures we can put into the bonus structure: we start from the simple bonus structure where the bank pays out a percentage of the trader’s profit. As this type of compensation does not have much control on the risks the trader could take, there is a need to turn to bonus structures which depend on the risks taken by the trader over the year: we

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*Because it makes sense and it makes things more simple, we are going to assume throughout the paper that the trader tries to maximize his expected bonus. For a trader with a different utility function, the problem can still be tackled within our framework.*
incorporate the Sharpe ratio of the trader’s trading account in the bonus structure. To make our model more realistic we put the trader’s skill into the equation as well. Finally we look at the consequences of adding a feature in the contract which fires the trader if his trading account goes down to some prescribed value. In the second part of the article, we concentrate on the horizon problem: the bank has a long-term horizon whereas the trader might have short term plans! We will try to see how the bank can make the trader’s horizon match with the bank’s horizon by making the bonus structure path-dependent.

2. Various Bonus Structures

2.1. Simple bonus structure

In this section, we are going to study the following bonus structure: at the end of the year, the trader gets a percentage of his profit

$$\text{bonus} = \lambda \max(\pi_T, 0)$$

where $\lambda$ is the percentage fixed in advance, and $T = 1$. This bonus structure is basic and we are going to explain our mathematical framework in details.

The trader’s bonus can be seen as a call option on his trading account: if the trading account is positive at the end of the year, then he chooses to exercise his option and gets his bonus. On the other hand, if he has performed badly and has lost money, then nothing happens: he does not get any bonus. Let’s call $V(t, S, \pi)$ the expected value of the bonus. Note that $V$ does not depend on the paths $S(t)$ and $q(\cdot)$ because Eqs. (1) and (2) show that the trading account $\pi$ is Markovian in the state variables $S, \pi$ and $q$; since the bonus structure does not depend on the paths of $S$ and $\pi$, neither may $V$ and the optimal strategy $q^*$ (see [4]).

Let $V^q(t, S, \pi)$ denote the value corresponding to the trading strategy $q(\cdot)$, where $q$ does not need to be optimal. The expected bonus is given by

$$V^q(t, S, \pi) = E[\exp^{-r(T-t)} \lambda \max(\pi_T, 0)]$$

where the expectation is taken under the physical measure. Now we use the fact that the trader chooses the trading strategy which maximizes his expected bonus

$$V(t, S, \pi) = \max_{|q| \leq C} V^q(t, S, \pi),$$

where $V$ is the maximal expected bonus. So

$$V(t, S, \pi) = \max_{|q| \leq C} E[\exp^{-r(T-t)} \lambda \max(\pi_T, 0)].$$

For the above stochastic control problem, Bellman’s principle of dynamic programming (see [3]) is formulated as:

$$V(t, S, \pi) = \max_{|q| \leq C} E[\exp^{-r\delta t} V(t + \delta t, S(t + \delta t), \pi(t + \delta t))]$$
where the expectation is maximized by proceeding optimally over the short time interval \([t, t + \delta t]\). This principle states that we find the optimal strategy at a given time by assuming that we already know the optimal strategy at all future times: in other words, that backward induction works. Bellman’s principle can be used to derive a PDE for \(V\), the Hamilton-Jacobi-Bellman (HJB) equation (see [3]):

\[
-V_t = \mu SV_S + r\pi V_\pi - rV + \frac{\sigma^2 S^2}{2} V_{SS} + \max_{|q| \leq C} \left[ q\left(\mu - r\right)V_\pi + \sigma^2 S^2 V_{\pi S} \right] + \frac{\sigma^2 S^2}{2} V_{\pi \pi} \tag{8}
\]

with final condition

\[
V(T, S, \pi) = \lambda \max(\pi, 0). \tag{9}
\]

To derive this equation, we have used Itô’s lemma as well:

\[
dV = V_t dt + \mu SV_S dt + \left(\pi + q(S(\mu - r)V_\pi) dt + \sigma^2 S^2 \left(\frac{1}{2} V_{SS} + qV_{\pi S} + \frac{q^2}{2} V_{\pi \pi}\right) dt. \tag{10}
\]

The HJB equation defines the optimal strategy \(q^*\) which is the maximiser of the term in brackets. Once this strategy is found, the PDE for \(V\) is defined. The verification result for the HJB equation, which demonstrates that the value \(V(\cdot)\) is attainable with some strategy and cannot be exceeded, follows a standard derivation (see [7]).

Throughout the paper, we assume \(C = 1\). We do not lose generality because \(V_{C=1}(t, S, \pi) = V_{C=1}(t, C^* S, \pi)\).

Now, the HJB equation can be simplified thanks to the symmetry of the problem; indeed, if we introduce the variable \(z = \frac{\pi}{S}\), then \(V(t, S, \pi) = S\phi(t, z)\), where \(\phi\) satisfies

\[
-\phi_t = (\mu - r)\phi - (\mu - r)z\phi_z + \max_{|q| \leq 1} \left[ q(\mu - r)\phi_z + \frac{\sigma^2}{2} (z - q)^2 \phi_{zz}\right] \tag{11}
\]

with the final condition \(\phi(T, z) = \lambda \max(z, 0)\).

Note that similar equations arise in the pricing of passport options (see [1]).

2.1.1. Numerical results

We use an explicit finite difference scheme to solve the PDE (see [8]) for explanation of these methods in finance. For each grid point, we find the optimal strategy \(q^*\) by checking the three candidates: \(q_1 = -1\), \(q_2 = 1\), and \(q_3\), which is the value for which the \(q\)-derivative of the term to maximize vanishes. Once the optimal strategy is found, we perform standard simulations of the asset price; we simulate the corresponding optimal trading strategy and we obtain histograms of the trader’s bonus and of the bank’s profit. We assume that the trader trades approximately once a day.
The End-of-The-Year Bonus: How to Optimally Reward a Trader? 283

In the following numerical examples, \( T - t = 1, S = 50, \mu = 0.1, \sigma = 0.2, \)
\( r = 0.0, \lambda = 0.1 \) and \( \delta t \), the time between two consecutive trades, is 0.004. So we
look at the trader’s and bank’s profit in the case where the trader is allowed to
trade a stock whose value is 50, drift is 10\%, volatility is 20\%, interest rate is zero,
and the trader will get 10\% of his eventual profit at the end of the year. We perform
100000 Monte-Carlo simulations of the trader following the strategy \( q^* \).

The optimal strategy \( q^* \) is always equal to one: in our example, the buy and
hold strategy is optimal.

In Fig. 1(a), we plot the histogram of the profit made by the trader: the proba-
bility of the trader receiving no bonus at the end of the year is 0.3459. His expected
bonus is 0.7344 and the standard deviation of his bonus is 0.8938.

In Fig. 1(b), we plot the histogram of the profit made by the bank. Its expected
profit is 4.4436 and the standard deviation of its profit is 10.4638.

Figure 1(c) is the histogram of \( (1) \), the trading account at the end of the year.
It is the sum of the bank’s profit and of the trader’s profit. As the optimal strategy
is equal to one and \( r = 0 \), we have \( \pi(1) = S(1) - S(0) \). This explains why the
distribution of \( \pi(1) \) is lognormal.

Finally, in Fig. 1(d), we plot the mean (+) and the standard deviation (o) of
the bank’s profit as a function of \( \lambda \). The greater \( \lambda \), the more the bank has to pay
out at the end of the year.

2.2. Bonus depending on the Sharpe ratio

Things get more interesting and perhaps more sensible, if the bonus also depends
on the realised Sharpe ratio. Indeed, the bank can now reward the trader according
to the profit he has made and to the risk he has taken to obtain this profit. It is
clear that for the bank, the ideal trader would make a large profit by taking very
little risk. So we introduce the new variable \( I \), the variance of the trading account.
It evolves according to

\[
dI = q^2 \sigma^2 S^2 dt
\]

so

\[
I = \int_0^t q^2 \sigma^2 S^2 dt.
\]

At time \( t = 0 \), \( \pi = I = 0 \) and the trader begins to trade the underlying asset.
At the end of the year, the bank gives the trader a bonus depending on the profit
made, \( \pi(T) \) and the Sharpe ratio

\[
\frac{\pi}{\sqrt{I}}
\]

We are ignoring the risk-free interest rate that should be in the Sharpe ratio.

Again,

\[
\max_{|q| \leq 1} (dV - rV dt) = 0.
\]
The equation for $V(t, S, \pi, I)$ is therefore

$$V_t + \mu SV_S + \sigma^2 S^2 V_{SS} + r \pi V_\pi - r V + \frac{\sigma^2 S^2}{2} V_{SS}$$

$$+ \max_{|q| \leq 1} \left[ q(S(\mu - r)V_\pi + \sigma^2 S^2 V_{\pi S}) + q^2 \left( \frac{\sigma^2 S^2}{2} V_{\pi \pi} + \sigma^2 S^2 V_I \right) \right] = 0 \quad (16)$$
with final condition

\[ V(T, S, \pi, I) = \max(\pi, 0)P\left(\frac{\pi}{\sqrt{I}}\right). \]  

(17)

A suitable form for the function \( P \) would be monotonically increasing from zero to a constant \( c < 1 \).
Again, a similarity solution is available: if we introduce the variables $z_1 = \frac{S}{S^*}$ and $z_2 = \frac{I}{S^*}$, then

$$V(t, S, \pi, I) = S\phi(t, z_1, z_2),$$

(18)

where $\phi$ satisfies

$$\frac{\partial \phi}{\partial t} + (\mu - r)\phi - (\mu - r)z_1\phi_{z_1}
+ \max_{|q| \leq 1} \left[ q(\mu - r)\phi_{z_1} + q^2\sigma^2\phi_{z_2} + \frac{\sigma^2}{2}(z_1 - q)^2\phi_{z_1z_1} \right] = 0$$

(19)

with final condition

$$\phi(T, z_1, z_2) = \max(z_1, 0)P\left(\frac{z_1}{\sqrt{2}}\right).$$

(20)

2.2.1. Numerical results

In the following example, we take the same assumptions as in the previous section. The only difference lies in the bonus payoff: we choose $P(x) = 0.2\frac{x}{\sqrt{x^2}}$. This function is a realistic example of a monotonically increasing function from 0 to 0.2.

In Fig. 2(a), we plot the bonus structure as a function of $\pi$ and $\frac{\pi}{\sqrt{\pi^2}}$.

![Fig. 2. Bonus structure with the Sharpe ratio and subsequent values of $\phi$ and $q^*$. (a) Trader’s bonus as a function of $\pi$ and $\frac{\pi}{\sqrt{\pi^2}}$. (b) Value of $\phi$ as a function of $z_1$ and $z_2$ and (c) The trading strategy $q$ at time $t = T - 1$ which maximises the expected bonus.](image-url)
Fig. 2. (Continued)
In Fig. 2(b), we plot $\phi$ as a function of $z_1$ and $z_2$.

In Fig. 2(c), we plot the optimal trading strategy $q^*$ as a function of $z_1$ and $z_2$. When $z_1$ is negative, the trading account is negative and the optimal strategy is 1: the trader has to take risks if he wants to maximize his expected bonus. On the
other hand, the optimal strategy is no longer 1 when the trading account is positive: this reflects the fact that the bonus structure depends on the Sharpe ratio.

In Figs. 3(a), 3(b), 3(c) and 3(d), we plot simulations of $S, q^*, \pi$ and $\sqrt{\eta}$ respectively. The trader is quite lucky in the first six months and his Sharpe ratio is 2.5, which is very good. At this point, if the Sharpe ratio increases, the expected bonus would not increase by much. And if the Sharpe ratio decreases, the expected bonus
would be comparatively more affected. This is why it is not worth taking too much risk. The reason for that is the choice of function $P$.

In Fig. 4(a), we plot the histogram of the trader’s bonus. The probability of the trader receiving no bonus at the end of the year is 0.3461. His expected bonus is 0.7113 and the standard deviation of the bonus is 1.1577.

Fig. 4. Various histograms when the Sharpe ratio is in the bonus structure. Histogram of the (a) trader’s bonus; (b) bank’s profit; (c) $\pi(T)$ and (d) Sharpe ratio.
In Fig. 4(b), we plot the histogram of the bank’s profit. Its mean is 4.3784 and its standard deviation is 9.6277.

Figure 4(c) is the histogram of \(\pi(1)\). This time the optimal strategy is not always equal to one and the distribution does not have to be lognormal anymore. This histogram need to be compared to the one of Fig. 1(c), where the bonus

![Histogram of bank's profit](c)

![Histogram of \(\pi(1)\)](d)

Fig. 4. (Continued)
structure does not depend on the Sharpe ratio. The histogram of Fig. 4(c) has a peak near 20 and its tail is smaller. This reflects the fact that if the trader starts the year well and manages to put his trading account above some critical value, then $q^*$ will become lower and the trading will stabilize around that critical value.

Finally, in Fig. 4(d), we plot the histogram of the Sharpe ratio. Its mean is 0.4399 and its standard deviation is 1.0611.

2.3. Skill

There is little point in rewarding traders, or even hiring them, if they do not possess some skill. Often “skill” is quantified by the Sharpe ratio. Here we want to suggest something a little more complex, but more realistic. We will model a possible way in which traders act, incorporating a skill factor that quantifies how much correct information they receive.

We are going to assume that our trader gets important and correct information about the direction of the market a fraction $p$ of the time. If the information is that the market will rise, he buys to the position limit, if the information is that the market will fall he sells to the same limit. The remaining $1 - p$ he trades according to the optimal strategy.

Let $p = \alpha\sqrt{d\tau}$ be the probability that the trader gets told the outcome. If he gets told, he trades up to the position limit. We get

$$d\pi = \begin{cases} r\pi dt + q(dS - rSdt) & \text{with probability } 1 - p; \\ r\pi dt + C|dS - rSdt| & \text{with probability } p. \end{cases}$$

This leads to

$$E\left[\frac{d\pi}{dt}\right] = r\pi - q\pi S + q\mu S + \alpha C\sigma S\sqrt{\frac{2}{\pi}}. \quad (21)$$

The PDE for $V$ is therefore

$$V_t + \mu SV_S + (r\pi + \alpha\sigma S\omega)V_z - rV + \frac{\sigma^2 S^2}{2}V_{zz}$$

$$+ \max_{|q| \leq 1} \left[ q(S(\mu - r)V_a + \sigma^2 S^2V_{aa}) + q^2 \left( \frac{\sigma^2 S^2}{2}V_{aa} + \sigma^2 S^2V_{zz} \right) \right] = 0 \quad (22)$$

where $\omega = \sqrt{\frac{2}{\pi}} = 0.7978846\ldots$, with the “other” $\pi = 3.1415926\ldots$.

The similarity solution $\phi$ satisfies

$$-\phi_t = (\mu - r)\phi - (\mu - r)z_1\phi_{z_1} + \alpha\sigma\omega\phi_{z_1}$$

$$+ \max_{|q| \leq 1} \left[ q(\mu - r)\phi_{z_1} + q^2 \sigma^2 \phi_{z_2} + \frac{\sigma^2}{2}(z_1 - q)^2 \phi_{z_1 z_1} \right], \quad (23)$$

with the usual final condition (23).
2.3.1. Numerical results

Given $\alpha$, we can calculate the means and standard deviations of the profits made by the bank and the trader via Monte-Carlo simulations. We perform 100000 MC simulations.

Fig. 5. Simulation of $S$ and subsequent values of $q^*$, $\pi$ and $\frac{\pi}{\sqrt{T}}$ when the trader’s skill factor is put in the model. Simulation of the (a) asset price; (b) optimal strategy; (c) trading account and (d) Sharpe ratio.
In the following examples, \( \alpha = 0.5 \) and \( \delta t = 0.004 \). So \( p = 0.03162 \ldots \).

At each time step, there is a probability of \( p \) for the trader to know the direction of the next price change.

In Figs. 5(a), 5(b), 5(c) and 5(d), we plot simulations of \( S, q^*, \pi \) and \( \frac{\pi}{\sqrt{t}} \) respectively. Note that \( q^* \) jumps to \(-1\) when the trader gets told that the asset price is going to go down.
In Fig. 6(a), we plot the histogram of the trader’s bonus. The probability of the trader receiving no bonus is 0.2133. The expected profit of the trader is 1.1311 and the standard deviation of his profit is 1.3986.

Fig. 6. Histograms when the trader’s skill factor is added in the model. Histogram of the (a) trader’s bonus and (b) bank’s profit.
In Fig. 6(b), we plot the histogram of the bank’s profit. The mean is 7.7908 and the standard deviation is 9.0146. Note that both histograms in Fig. 6 are compressed compared to the histograms of Fig. 4 (trader with no skill). The effect of the skill is to make the process sometimes deterministic; it is therefore not surprising to obtain histograms which are less spread out.

Fig. 7. Mean (+) and standard deviation (o) of the (a) trader’s profit and (b) bank’s profit as a function of the trader’s skill factor.
The End-of-The-Year Bonus: How to Optimally Reward a Trader?

In Fig. 7(a), we plot the mean (+) and the standard deviation (o) of the trader’s profit as a function of the skill factor $\alpha$. Both the mean and the standard deviation increase with $\alpha$.

In Fig. 7(b), we plot the mean (+) and the standard deviation (o) of the bank’s profit as a function of the skill factor $\alpha$. As $\alpha$ increases, the mean increases and the standard deviation decreases. This explains why the shape of the histogram of Fig. 6(b) is more concentrated than the one of Fig. 4(b).

A skilled trader is good both for the trader’s bonus and for the bank’s profit.

2.4. Firing

If $\pi$ gets negative, the optimal strategy for the trader becomes $q^* = 1$: the trader might be tempted to take very large risks in the hope of bringing $\pi$ back to a positive value. The bank does not want traders to take large risks; its policy should therefore discourage this type of behavior. For example, the bank could fire the trader if its trading account reaches a given negative value. This condition would discourage the trader to take large risks; indeed, if its trading account gets dangerously close to the red zone after a series of bad luck, he would get out of this zone using his skill, not his luck.

The PDE is as in Eq. (26), but we now have the new boundary condition

$$V(t, S, F, I) = 0$$

where $F$ is the negative value of the trading account for which the trader gets fired. This boundary condition actually means that if the trading account ever reaches $F$, then the trader does not get any bonus at the end of the year. So we suppose that these two perspectives are just as bad as each other from the trader’s point of view. If one thinks that the prospect of no bonus is not as bad as the prospect of getting fired, one can replace the boundary condition by $V(t, S, F, I) = G$, with $G$ a suitable negative function.

Note that no similarity solution is available and computation time is therefore much longer.

2.4.1. Numerical results

In the following numerical examples, we use the same inputs as in the previous section apart from $\delta t$: we use $\delta t = 0.008$.

Figure 8 shows the optimal strategy $q^*$ as a function of $\pi$ and $I$ ($S = 50$ and $F = -9$). Note that when the trading account is negative, the optimal strategy is not 1 anymore: this should discourage the trader from taking large risks.

In Fig. 9(a), we plot the mean (+) and the standard deviation (o) of the trader’s profit as a function of $F$. The trader would rather have no firing boundary condition!

In Fig. 9(b), we plot the mean and the standard deviation of the bank’s profit as a function of $F$. The firing boundary condition has a bad effect on the bank’s
expected profit. But it can be seen as an insurance against traders losing a lot of money.

In Table 1, we look at the probability for the trader to get fired for different values of \( F \).

### Table 1. Probability of getting fired for different \( F \)’s.

<table>
<thead>
<tr>
<th>( F )</th>
<th>15</th>
<th>12</th>
<th>9</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.013</td>
<td>0.0358</td>
<td>0.0845</td>
<td>0.1362</td>
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</tbody>
</table>

#### 3. Path-Dependent Bonus Structures

##### 3.1. High-water mark

Another issue in the bonus problem is that the bank has a long-term horizon in its investments whereas the trader might have a shorter horizon. Indeed, the trader is not going to work for the bank forever and could therefore trade according to a shorter time horizon. So there is a need for the bank to choose a bonus structure which takes this problem into account: a bonus structure which would push the trader to have a longer term horizon.
One way to ensure that the trader’s fate is matched with the bank’s is to include a “high-water mark” in the model. These types of features are used in hedge funds managers’ compensation contracts. This simply means that the bank keeps track of the accumulated trading account of the trader. For the trader to have a bonus, his accumulated profit (since time $t = 0$) must be positive, regardless how well he
has done over the year. Moreover, the bonus is calculated on the above water profit only. For example, if the trader brings his trading account from −1 to 3 over a year, his above water profit will be 3, not 4. These two conditions ensure that the trader’s main goal is to keep the accumulated profit above water.

In this section we drop the Sharpe ratio to keep things simple and the bonus structure is

\[
\text{bonus} = \begin{cases} 
\lambda \max(\pi, 0), & \text{if } \pi_N + e^\pi_{N-1} \geq 0; \\
0, & \text{otherwise}.
\end{cases}
\]

So Eq. (14) is satisfied between bonus times and at each time \( n = 1, \ldots, N - 1 \) (where \( N \) is the time horizon of the bank) there is a jump condition:

\[
V(n^-, S, \pi; \pi_{n-1}) = \begin{cases} 
V(n^+, S, 0; e^\pi_{n-1} + \pi) + \max(\pi, 0), & \text{if } \pi_n + e^\pi_{n-1} \geq 0; \\
V(n^+, S, 0; e^\pi_{n-1} + \pi), & \text{otherwise}.
\end{cases}
\]

Where \( n^- \) is the time just before time \( n \) and \( n^+ \) is the time just after time \( n \).

In the above equation, the parameter is the accumulated trading account. In terms of the similarity solution, the jump conditions are

\[
\phi(n^-, z; z_{n-1}) = \begin{cases} 
\phi(n^+, 0; e^\pi_{n-1} + z) + \max(z, 0), & \text{if } z_n + e^\pi_{n-1} \geq 0; \\
\phi(n^+, 0; e^\pi_{n-1} + z), & \text{otherwise}.
\end{cases}
\]

And the final condition is

\[
\phi(N, z; z_{N-1}) = \begin{cases} 
\lambda \max(z, 0), & \text{if } e^\pi_{n-1} + z \geq 0; \\
0, & \text{otherwise}.
\end{cases}
\]

3.1.1. Numerical results

In the following numerical example, we look at a two years period \( (N = 2) \), \( \lambda = 0.1 \), \( \mu = 0.1 \) and \( \alpha = 0 \).

In Fig. 10(a), we plot the histogram of the profit made by the trader. The mean is 1.3994 and the standard deviation is 1.4541.

In Fig. 10(b), we plot the histogram of the profit made by the bank. The mean is 9.9825 and the standard deviation is 14.2879.

3.2. Another path-dependent bonus structure

Another way to attract traders with a long-term horizon is to improve the bonus structure of good traders. More explicitly, the bank changes the function \( P \) to a greater function from one year to the next one if the trader performs well. In his first year the trader’s bonus is \( \max(\pi, 0)P_1(\frac{\lambda}{\sqrt{2}}) \). In his second year, his bonus structure is the same if he has not made money in his first year. However it is \( \max(\pi, 0)P_2(\frac{\lambda}{\sqrt{2}}) \) if he has made money in his first year, with \( P_2 \geq P_1 \). Similarly, for the third year he could move from \( P_1 \) to \( P_2 \), or from \( P_2 \) to \( P_3 \), with \( P_3 \geq P_2 \). However, if he loses money in his second year, he goes back down to \( P_1 \). We are
Fig. 10. Histograms in the high-water mark model. Histogram of the (a) trader’s bonus and (b) bank’s profit.
going to write the equations for this problem in order to find the optimal strategy $q^*$. Looking at the trader’s histogram profit and its own histogram of profit, the bank could then decide which $P_n$’s fit the best with its risk preferences.

As the expected bonus clearly depends on which bonus structure is “activated”, this makes the problem slightly more complicated. Indeed, we have now $N$ final conditions ($N$ is the time horizon of the bank):

$$V_i(N, S, \pi, I) = \max(\pi, 0)P_i \left( \frac{\pi}{\sqrt{T}} \right),$$  \hspace{1cm} (25)

for $i = 1, \ldots, N$. The parameter $i$ tells which bonus structure is activated. For example, if $i = N$, this means that the trader has made money every single year and his bonus at time $N$ will be $\max(\pi, 0)P_N(\frac{\pi}{\sqrt{T}})$. At each bonus time there will be jump conditions. At time $n$ there will be $n$ jump conditions:

$$V_i(n^-, S, \pi, I) = \begin{cases} V_{i+1}(n^+, S, 0, 0) + \pi P_i \left( \frac{\pi}{\sqrt{T}} \right), & \text{if } \pi \text{ is positive;} \\ V_i(n^+, S, 0, 0), & \text{if } \pi \text{ is negative}, \end{cases}$$

for $i = 1, \ldots, n$. There are only $n$ jump conditions at time $n$ because the trader cannot do better than making money every year…

The HJB Eq. (19) is satisfied between the bonus times and our stochastic control problem is well-defined. Note as well that the dimension of the problem can be reduced via similarity solutions. The jump conditions are then

$$\phi_i(n^-, z_1, z_2) = \begin{cases} \phi_{i+1}(n^+, 0, 0) + z_1 P_i \left( \frac{z_1}{\sqrt{z_2}} \right), & \text{if } z_1 \text{ is positive;} \\ \phi_i(n^+, 0, 0), & \text{if } z_1 \text{ is negative}. \end{cases}$$

The HJB is as in (22) and the final conditions are

$$\phi_i(N, z_1, z_2) = \max(z_1, 0)P_i \left( \frac{z_1}{\sqrt{z_2}} \right),$$  \hspace{1cm} (26)

3.2.1. **Numerical results**

In the following numerical example, we take $N = 2$ and $P_i(x) = \lambda_i x^2$. In Fig. 11(a), we plot the trader’s expected profit as a function of $\lambda_1$ and $\lambda_2$. In Fig. 11(b), we plot the standard deviation of the trader’s profit as a function of $\lambda_1$ and $\lambda_2$. In Fig. 11(c), we plot the bank’s expected profit as a function of $\lambda_1$ and $\lambda_2$. In Fig. 11(d), we plot the standard deviation of the bank’s profit as a function of $\lambda_1$ and $\lambda_2$. 

4. Conclusion

The end-of-the-year bonus is a source of motivation for the trader throughout the year and a way for the bank to keep its good traders. However a badly designed
The End-of-The-Year Bonus: How to Optimally Reward a Trader?

Fig. 11. Trader’s bonus and bank’s profit as a function of $\lambda_1$ and $\lambda_2$. (a) Trader’s expected profit; (b) standard deviation of the trader’s profit; (c) bank’s expected profit and (d) standard deviation of the bank’s profit.
Fig. 11. (Continued)
bonus structure could have undesired effects on the trader’s trading behavior. Indeed, the trader could be tempted to take large risks or he could trade with a short-term view. For example, the simple bonus structure which pays out a percentage of the profit encourages the trader to be long the asset up to the position limit. There is therefore a need to develop quantitative methods so that the bank can adopt a bonus policy which would fit with its risk preferences.

In this article we have built a mathematical framework for the study of this problem. We have studied various bonus structures of practical interest. We found the trader’s optimal trading strategy by solving HJB equations. In our numerical examples, we have looked at the effect of the bonus on the optimal strategy, on the distribution of the trader’s profit and on the distribution of the bank’s profit. We have found that the bonus depending on the Sharpe ratio stops the optimal strategy from being equal to the position limit in the case where the trading account is positive. Our solution for tackling the case where the trading account is negative is to add a feature in the contract which says that the trader gets fired if his trading account falls to a prescribed value. Our mathematical framework allows us to model the trader’s skill as well. The skill is quantified by a skill factor that represents how much correct information the trader receives. Its effect is to add an advection term in the HJB equation. The solution that we have proposed in order to encourage the trader to have a long-term horizon is to make the bonus structure path-dependent. We have studied two such models.

Bonus compensation of traders is a very important issue: there are numerous examples of traders taking a lot of risk in the hope of a higher bonus, and losing a lot of money. Yet no quantitative methods had been developed to tackle this problem. In this paper, we have built a mathematical framework for the study of the bonus problem and we have determined the features that a good bonus structure should depend on.

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