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On the occasion of Walter Schachermayer's 70th birthday: the mathematics of arbitrage

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1 Introduction

Born on July 24, 1950 Walter Schachermayer lived his childhood in Linz, Austria. He attended the Akademisches Gymnasium in Linz, where also Ludwig Boltzmann learned Latin, Greek, Physics and Mathematics about 100 years before him. Walter studied Mathematics, Economics and Computer Science in Vienna and completed his PhD under Johann Cigler's supervision on "Cylindrical measures and the Radon-Nikodym-property of Banach spaces" in 1976. He chose this PhD topic inspired by Johann Cigler's unique lectures, in particular his presentation of the theory of distributions. This experience opened his eyes for the true nature of mathematics, as Walter describes it himself, and stimulated him to study a year in Paris where he attended the Séminaire Maurey-Schwartz. After finishing his PhD he continued his career in Clermont-Ferrand, Mexico City, Linz, Vienna and Paris and is now Professor Emeritus of Mathematics at the University of Vienna.

Walter has received many prizes, among which the prestigious Wittgenstein Award from the Austrian Science Foundation in 1998 and an advanced ERC grant 2009 are outstanding. He received honorary doctorates from the Université Paris-Dauphine in 2011 and Universidad de Murcia in 2018. He is member of the Leopoldina and Academia Europaea. In 2016 he was awarded the prize for Natural Sciences of the city of Vienna. He has supervised and mentored a great number of students and post-docs in Vienna, including the authors of this article. For many of them the years spent in Walter's group had a lasting influence to pursue

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a scientific career, which was often crowned by professorships in distinguished universities worldwide.

Several ground-breaking results in mathematical finance are inseparably connected with his name. They paved the way towards today's mathematical understanding of arbitrage [24, 9, 10, 11, 12, 1], utility optimization [19, 20], transactions costs [25, 14, 6] or martingale optimal transport [2], to name only a few.

In his mathematical reasoning abstract functional analysis and stochastic analysis meet in a unique way with questions from finance and Viennese charm. Unforgettable are talks at conferences where he illustrates most abstract functional analytic results with the simplest possible example. He is also known as a smiling advocat of mathematical rigor in every detail, which is the only way to create results which are still true tomorrow.

Walter is also famous for eloquent, sharp and clear contributions to public discussions in Austria: he demonstrated impressively the salient difference between "possible" and "likely" on the example of the (probabilistically) pointless rerun of the last presidential election. He also contributed to discussions on public debts of European countries by pointing to the fact that every debitor has to have a creditor as counterpart, thus presenting questions about goverment's debts in the proper light.

To exemplifiy Walter's approach to mathematical problems we shall focus on a preeminent contribution, a joint work with Freddy Delbaen in Mathematische Annalen, 1994, see [9]. Needless to say that our guided tour through this impressive work would also qualify as a laudatio for Freddy Delbaen. In this article they solved an open problem in the foundations of mathematical finance, namely how to characterize the absence of arbitrage in an economically convincing way and how to fully establish its workable probabilistic counterpart. The result itself requires, although clearly rooted in financial practice, a deep understanding of stochastic and functional analysis, which brought the theory of semimartingales as well as geometry of Banach spaces in the arena of finance: an instance of the maxim "there is nothing more practical than a good theory".

Let us first state the result in non-formal terms. The question which models are adequate to describe prices in financial markets has many layers of answers. Should models be deterministic or stochastic, chosen by methods from partial differential equations, statistics, econometrics or economics, should they be analytically tractable, or rather robust? Within all these categories profound answers can be formulated, but the most far reaching answer, solving the problem in utmost generality, can be given when prices are modeled by *continuous time* stochastic processes. This is the *Fundamental Theorem of Asset Pricing* (FTAP).

To illustrate the key idea in a simple setup, imagine a *discrete time* model for the next time instant in a financial market with d assets. This is a model, which has known prices $S_0 \in \mathbb{R}^d$ and random prices S_1 in the next time step. If we want to compare prices, we should quote them in discounted terms, that is relative to some riskfree, i.e. non-defaultable quantity (for instance an Austrian bond). An investment is just a choice $\phi \in \mathbb{R}^d$, where each component of the investment vector ϕ corresponds to the number of shares held in the respective asset. Apparently the value of this investment now is $\langle \phi, S_0 \rangle$, in the next instant $\langle \phi, S_1 \rangle$, its change (by virtue of being discounted) is just $\langle \phi, S_1 - S_0 \rangle$. Such an investment should not produce a riskless gain, a reasonable economic assumption called *Absence of Arbitrage*. A riskless gain or arbitrage just means that with probability 1, we do not loose anything and with positive probability we gain something. *Absence of Arbitrage* can therefore be formulated as follows: there does not exist $\phi \in \mathbb{R}^d$ such that

$$0 \neq \langle \phi, S_1 - S_0 \rangle \geq 0$$

almost surely. Turning to mathematics, this implies that S_0 must lie in the relative interior of the convex hull of the support of the law of S_1. Indeed, otherwise 0 would not lie in the relative interior of the convex hull of the support of the law of $S_1 - S_0$, whence a separating hyperplane provides us with a vector ϕ, an arbitrage.

Recall now the non-trivial result that the expectation of a random variable X with values in $\mathbb{R}^d$ actually lies in the relative interior of the convex hull of the support of its law (sharpening the easy result that it lies in the closed convex hull) and that every point in the relative interior of the convex hull of the support of the law is an expectation of X with respect to some equivalent measure. This together with the *Absence of Arbitrage condition* then implies the existence of a *martingale measure*, i.e. $Q \sim P$ such that $E_Q[S_1] = S_0$. The simpler converse direction then yields equivalence between these two properties. Note that for this and the subsequent analysis the set of nullsets is always fixed but not necessarily the measure P. Relaxing this assumption led to an important strand of research, *robust finance*, to which Walter contributed with several co-authors. We refer in particular to a model-free version of the FTAP, see [1], with Beatrice Acciaio, Mathias Beiglböck and Friedrich Penkner.

Going from the one period case towards a dynamic picture, namely making S_0 a random variable itself measurable with respect to today's information (modeled by a σ-algebra), is a bit delicate. Either one goes for a conditional version of the previous argument, which of course exists, or one brings in a new aspect: *duality*. This also reveals the actual nature of the measure Q. Consider the set of outcomes of investments C_0 at zero initial wealth, i.e. the set of all $f = \langle \phi, S_1 - S_0 \rangle - g$ where ϕ is now a random variable measurable with respect to the initial information and $g \geq 0$ corresponds to consumption. Then, clearly, every martingale measure Q yields $E_Q[f] \leq 0$ with the understanding that $0 \leq E_Q[g] \leq \infty$. Absence of arbitrage just means that

$$C_0 \cap L^0_{\geq 0} = \{0\},$$

where L^0 denotes the space of (equivalence classes of) all random variables. If C_0 is appropriately closed, then Q can just be understood as a strictly positive element of the dual cone of C_0. Of course C_0 is closed in probability, which actually is enough to guarantee the existence of a strictly positive dual element. This now constitutes on a one period level the assertion of the fundamental theorem of asset pricing. The question is whether there is a continuous time version of this argument.

When we see these arguments, several problems in view of a fully time-continuous theorem become apparent:

- Since we do not want to impose artificial moment conditions on discounted prices, we have to work in L^0 with respect to convergence in probability. Even though being a topological vector space, L^0 in general is not locally convex and generically its dual space is just $\{0\}$. This points towards some problems when speaking about polar cones and duality, to say the least.
- Even if we are able to work with an appropriate duality, the polar cone will generically *not* be generated a single measure Q but have multiple dimensions. This means when passing from a one period level to continuous time one has to concatenate not-uniquely given measures Q, which are conditionally dual elements. This points towards the use of measurable selection theorems, which is often delicate and cumbersome, and a road which has not been taken by Freddy Delbaen and Walter Schachermayer.
- They rather first revealed the nature of discounted price processes, namely that they have to be semimartingales. This in turn led to intricate questions in stochastic analysis, precisely how to define and analyze spaces of terminal values of stochastic integrals.

It is the goal of this article to present Walter's and Freddy's solution to *all* the above problems. Before doing so, let us put their result, which can be seen as the single most important result of mathematical finance, in a historical context.

2 Some historical remarks on the FTAP

Today's most cited FTAP version proved by Freddy and Walter establishes in continuous time under a fairly weak assumption on a set $\mathcal{X}$ of admissible portfolio wealth processes for self-financing, discounted portfolios, a property called *No Free Lunch with Vanishing Risk* (NFLVR), the existence of an *equivalent separating measure* $Q \sim P$. This rather technical sounding assertion is the correct and sharp mathematical formulation of the vague "meta-theorem" stating that no arbitrage is *essentially* equivalent to the existence of an equivalent martingale measure (as sketched in the one period model above) and has thus tremendous consequences: first, models can be easily characterized to satisfy (NFLVR) by simply checking whether such a separating measure $Q \sim P$ exists. Second, the statement of FTAP is, mathematically speaking, the characterization of typical elements of a polar cone, which in turn allows to look at optimization problems from a dual point of view. Third, by simple economic arguments, separating measures $Q \sim P$ lead to pricing structures for general payoffs.

The long history of FTAP is widely ramified, as can be seen from the excellent overview article [26] by Walter Schachermayer or the monograph "The Mathematics of Arbitrage" [12] by Freddy Delbaen and Walter Schachermayer. Let us here only briefly state the main milestones up to 1998. For further developments after 1998 we refer to [5] and the references therein. The subsequent presentation is also based to a large extent on this article.

The history of FTAP traces back to the work of Fisher Black and Myron Samuel Scholes [4] as well as Robert Merton in 1973. Indeed, their formula was the starting point for an investigation between the relation of pricing by no arbitrage considerations and pricing by taking "risk neutral" expectations (with respect to a martingale measure). In the late 1970s and early 1980s major advances in establishing a precise mathematical connection between those notions and proving first versions of FTAP in different settings were achieved by Stephen Ross [23], Michael Harrison, David Kreps and Stan Pliska [15, 16, 21]. These seminal papers have been generalized and further developed in many directions, in particular a first complete proof of FTAP in finite discrete time was given by Robert Dalang, Andrew Morton and Walter Willinger [7] extending the Harrison-Pliska result [16]. In continuous time, Christophe Stricker [27] combined the result of David Kreps with a theorem by Jia-An Yan [28], which is now known under the name Kreps-Yan theorem and which states the equivalence between *No free lunch* (NFL) and the existence of an equivalent separating measure (see Theorem 2).

The remaining major challenge was to replace the strong condition of (NFL) (involving closures in the weak-$*$-topology in L^∞) by an economically convincing concept which only slightly strengthens the intuitive notion of absence of arbitrage. It turns out that the concept of (NFLVR) introduced by Freddy Delbaen and Walter Schachermayer in [9] is precisely the right minimal and economically meaningful requirement which still allows to conclude the existence of an equivalent separating measure. Also the concept of *No Free Lunch with Bounded Risk*, as applied by Walter Schachermayer in [24] in the discrete infinite time horizon case, would serve this purpose, however, (NFLVR) is even weaker. Freddy and Walter consider as *set of admissible portfolio wealth processes $\mathcal{X}$* stochastic integrals $(\phi \bullet S)$, for admissible integrands ϕ with respect to a one dimensional locally bounded semimartingale S. Their beautiful and impressive proof builds on deep insights and is in some parts quite tricky. It was taken up by Youri Kabanov who introduced, inspired also by [8], in a sharply focused paper [17], an abstract setting of admissible portfolio wealth processes (see Remark 1 below) allowing for convexity constraints and unbounded jumps: Youri Kabanov's insight was that the proof of [9] transfers almost literally to this novel setup. This is another illustration of the statement, that Freddy Delbaen and Walter Schachermayer have chosen the simplest setting which contains all aspects and the complete proof of the most general finite dimensional case. While cleverly working in a one dimensional setting for this general version of the FTAP, Walter considered more or less at the same time with his first PhD student Irene Klein the setting of large financial markets [18], which substantially differs from the finite dimensional case. Finally in [11], Freddy and Walter considered the extension to unbounded continuous time stochastic processes, where the relation between a separating measure and a (generalized) martingale measure is more subtle, which has independently been proved in [17], too.

3 The setting of the proof of FTAP

Cumulative gains and loss processes, which are ubiquitous in finance, appear as discretizations of integrals. Hence it is completely natural to assume that discounted price processes are actually *good integrators*, i.e. stochastic processes where cumulative gains and loss processes satisfy a certain continuity property. It is a deep result, the Bichteler-Dellacherie theorem, that good integrators are actually semimartingales, i.e. the sum of a local martingale and an adapted process of finite total variation. Walter Schachermayer also contributed to this topic by providing rather recently together with Mathias Beiglböck and Bezirgen Veliyev another elementary proof of this important result under an even weaker and financially inspired assumption (see [3]).

Let now $\mathbb{S}$ be the space of such good integrators, i.e. semimartingales X defined on a finite interval $[0,1]$ and starting from zero. The space $\mathbb{S}$ is equipped with the Emery topology, named after Michel Emery and defined by the metric

$$d_E(X_1,X_2) := \sup_{K \in b\mathcal{E},\, \|K\|_\infty \leq 1} E\left[|(K \bullet (X_1 - X_2))|_1^* \wedge 1 \right],$$

where $|X|_1^* = \sup_{t \leq 1} |X_t|$, $b\mathcal{E}$ denotes the set of simple predictable strategies, that is, K is of the form

$$K = \sum_{i=0}^{n} K_i 1_{]\tau_i, \tau_{i+1}]},$$

with $n \in \mathbb{N}$, stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \tau_{n+1} = 1$ and K_i are $\mathcal{F}_{\tau_i}$-measurable random variables.

The space of semimartingales is a complete topological vector space with the Emery topology, which follows essentially from the Bichteler-Dellacherie Theorem, see [13].

Beside convergence in the Emery topology, pathwise uniform convergence in probability plays an essential role. This type of convergence is metrized by

$$E[|X-Y|_1^* \wedge 1] = d(X,Y)\,,$$

which makes the space of càdlàg processes a complete topological vector space. Obviously uniform convergence in probability is a weaker topology than the Emery topology.

Let now S be a one dimensional semimartingale. Then the set $\mathcal{X}$ of all stochastic integrals $(\phi \bullet S)$, where ϕ is S-integrable such that there exists a uniform bound from below $(\phi \bullet S) \geq -\lambda$, for some $\lambda \geq 0$, is a set of admissible wealth processes generated by

$$\mathcal{X}_1 := \big\{ (\phi \bullet S) \mid \phi \text{ is } S\text{-integrable and} \, (\phi \bullet S) \geq -1 \,\big\} \tag{8}$$

in the sense that $\mathcal{X} = \cup_{\lambda>0} \lambda \mathcal{X}_1$.

Remark 1. *Youri Kabanov suggests in [17] to work in the following more general setting: we are just given a convex set $\mathcal{X}_1 \subset \mathbb{S}$ of semimartingales, which is supposed to satisfy the following axiomatic properties*

- *starting at* 0*,*
- *bounded from below by* -1*,*
- *being closed in the Emery topology, and*
- *the* concatenation property*: for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}_1$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1$.*

Note that the set (8) satisfies precisely the properties stated in Remark 1. Convexity and the concatenation property are both just facts of stochastic integration theory, while the most crucial property namely closedness in the Emery topology is a consequence of Jean Mémin's theorem (see [22]).

We denote by $\mathcal{X}$ the set $\mathcal{X} = \cup_{\lambda>0} \lambda \mathcal{X}_1$ and call its elements *admissible portfolio wealth processes*. The elements of $\mathcal{X}_1$ are called 1-admissible wealth processes. We denote by K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}_1$, at terminal time $T = 1$.

Let us introduce several notions of absence of arbitrage, for which we define the following convex cones:

$$C_0 := K_0 - L^0_{\geq 0}, \quad C := (K_0 - L^0_{\geq 0}) \cap L^\infty. \tag{9}$$

(NA) The set $\mathcal{X}$ is said to satisfy *No Arbitrage*, if

$$(K_0 - L^0_{\geq 0}) \cap L^0_{\geq 0} = C_0 \cap L^0_{\geq 0} = \{0\},$$

which can be easily shown to be equivalent to

$$((K_0 - L^0_{\geq 0}) \cap L^\infty) \cap L^\infty_{\geq 0} = C \cap L^\infty_{\geq 0} = \{0\}.$$

(NFLVR) The set $\mathcal{X}$ is said to satisfy *No Free Lunch with Vanishing Risk*, if

$$\overline{C} \cap L^{\infty}_{\geq 0} = \{0\},$$

where $\overline{C}$ denotes the norm closure in L^{∞}.

(NFL) The set $\mathcal{X}$ is said to satisfy *No Free Lunch*, if

$$\overline{C}^{*} \cap L^{\infty}_{\geq 0} = \{0\},$$

where $\overline{C}^{*}$ denotes the weak-$*$-closure in L^{∞}.

(NUPBR) The set $\mathcal{X}_1$ is said to satisfy *No Unbounded Profit with Bounded Risk*, if K_0^1 is a bounded subset of L^0.

Remark 2. *1. (NFLVR) can be proved to be equivalent to (NA) and (NUPBR), i.e., (NFLVR) $\Leftrightarrow$ (NA) + (NUPBR) (see [9, Corollary 3.8]). This is an essential insight.*

2. (NFLVR) or even (NUPBR) are economically convincing minimal requirement for models, but only (NFL) allows to conclude relatively directly the existence of an equivalent separating measure, defined below.

Definition 1. *The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property, if there exists an equivalent measure $Q \sim P$ such that $\mathbb{E}_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.*

Under (NFL), the (ESM) property is a consequence of the Kreps-Yan Theorem, which in turn follows directly from Hahn-Banach's Theorem. For convenience we provide a proof here following [17]:

Theorem 2. *Fix $p \in [1, \infty]$ and set q conjugate to p. Suppose $C \subseteq L^p$ is a convex cone with $C \supseteq -L^p_{\geq 0}$ and $C \cap L^p_{\geq 0} = \{0\}$. If C is closed in $\sigma(L^p, L^q)$, then there exists $Q \sim P$ with $\frac{dQ}{dP} \in L^q(P)$ and $\mathbb{E}_Q[Y] \leq 0$ for all $Y \in C$.*

Proof. Any $x \in L^p_{\geq 0} \setminus \{0\}$ is disjoint from C, so we can apply the Hahn-Banach-theorem to strictly separate x from C by some $z_x \in L^q$. The cone property gives us $\mathbb{E}[z_x Y] \leq 0$, for all $Y \in C$ and $C \supset -L^p_{\geq 0}$ gives $z_x \geq 0$. Strict separation implies $z_x \neq 0$, so that we can normalize to $\mathbb{E}[z_x] = 1$.
We next form the family of sets $\{\Gamma_x := \{z_x > 0\} | x \in L^p_{\geq 0} \setminus \{0\}\}$. Then one can find a countable subfamily $(\Gamma_{x_i})_{i \in \mathbb{N}}$ with $P[\cup_i \Gamma_{x_i}] = 1$. For suitably chosen weights $\gamma_i > 0$, $i \in \mathbb{N}$, one gets that $Z := \sum_{i=1}^{\infty} \gamma_i z_{x_i}$ is $Z > 0$ almost surely with respect to P, $Z \in L^q$ and $\mathbb{E}[ZY] \leq 0$, for all $Y \in C$. Through normalization we get to $\mathbb{E}[Z] = 1$, then $dQ := ZdP$ does the job. □

Apparently we have

$$\text{(NFL)} \Longrightarrow \text{(NFLVR)} \Longrightarrow \text{(NA)},$$

but it is an astonishing and deep insight of Walter Schachermayer and Freddy Delbaen that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.

The fundamental theorem of asset pricing then reads as follows:

Theorem 3. *Under (NFLVR) the cone C is weak $*$-closed, hence (NFL) holds, which is equivalent to (ESM). In other words: (NVLVR) $\Leftrightarrow$ (ESM).*

4 A guided tour through the proof of FTAP

In this section we comment on the main steps of the proof of FTAP as presented in [9]. The proof actually splits into two parts: First a series of conclusions are presented, which can be easily motivated with financial (trading) arguments. Secondly five lemmas follow, whose content is more technical and which are considerably harder to prove.

The first series of conclusions is the following:

1. The convex cone C defined in (9) is closed with respect to the weak-$*$-topology, if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 uniformly bounded from below and converging almost surely to f it holds that $f \in C_0$, see [9, Theorem 2.1] essentially tracing back to A. Grothendieck. Notice that this step, whose core is the Krein-Smulian theorem, reduces the calculation of the weak-$*$-closure to a calculation with sequences.

2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.

3. By (NA) it follows that each $Y^n \in \mathcal{X}_1$ by a simple trading argument.

4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \operatorname{conv}(Y^n, Y^{n+1}, \dots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely. This is another appearance of Komlos lemma, which is a crucial tool in the proof.

5. This implies that the set $\widehat{K}_0^1 \cap \{g \in L_0 \,|\, g \geq f\}$, where $\widehat{K}_0^1$ denotes the closure of K_0^1 in L^0, is non-empty. Since it is also bounded by (NUPBR) and closed, a maximal element h_0 exists (see [9, Lemma 4.3]). Since $h_0 \in \widehat{K}_0^1$, we can find a sequence of semimartingales $X^n \in \mathcal{X}_1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

6. The previously constructed "maximal" sequence of semimartingales $X^n \in \mathcal{X}_1$ converges pathwise uniformly in probability, i.e. $|X^n - X|_1^* \to 0$ in probability, to some càdlàg process X (see [9, Lemma 4.5]). This is again a beautiful trading argument, where financial intuition meets topology.

Even though the processes X^n are just semimartingales, they behave in several respects like martingales, in particular when convergence of terminal values leads to uniform convergence in probability. This phenomenon is of course not true for finite variation processes. At this point one could conjecture that a sort of martingality with respect to an equivalent measure could hold, but it is not at all clear how to even formulate this.

Since it is of crucial importance we devote a proper definition to maximality as given before Lemma 4.3 in [9]:

Definition 2. *An element $h_0 \in \widehat{K}_0^1$ (where $\widehat{K}_0^1$ denotes the closure of elements of K_0^1 which dominate f) is called maximal, if it is maximal with respect to the pointwise (partial) ordering in L^0.*

It is now the goal to show that the sequence (X^n) constructed in 6. above converges to X in the Emery topology, an apparently much stronger statement. From this it follows that $h_0 = \lim_{n\to\infty} X_1^n = X_1 \in K_0^1$, since $\mathcal{X}_1$ *is closed in the Emery topology*. This in turn implies that $f \in C_0$, which finishes the proof by step (i) above.

Convergence in the Emery topology can be shown with respect to any equivalent measure $Q \sim P$, since this notion of convergence only depends on the equivalence class of probability measures. By the basic convergence result 6. we know that $\xi := \sup_n |X^n|_1^* \in L^0$ (after passing to a subsequence). We can therefore find a measure $Q \sim P$ (take, e.g., $dQ/dP = c\exp(-\xi)$) such that $X^n \in L^2(Q)$, hence we can continue the analysis with L^2-methods, in order to prove Emery convergence with respect to Q. This is an old trick, which works due to Bichteler-Dellacherie or Girsanov-Meyer for semimartingales.

Now the series of more technical lemmas starts: assume (NUPBR), take a sequence of (special thanks to an appropriate change of measure) semimartingales $X^n = A^n + M^n$ whose sup-processes $|X^n|_1^*$ are uniformly bounded in $L^2(Q)$.

1. First key lemma: the sequence $|M^n|_1^*$ is bounded in L^0 (see [9, Lemma 4.7]). Several trading arguments take place here. In general it is difficult to estimate the martingale part when just knowing that a sum with some total variation process converges uniformly in probability.

2. Second key lemma: define $\tau_c^n := \inf\{t \mid |M^n|_t^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty[} \bullet X^n)$, then for every $\varepsilon > 0$ there is $c_0 > 0$ such that for all

$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$$

it holds that $Q[|\widetilde{M}|_1^* > \varepsilon] \leq \varepsilon$ (see [9, Lemma 4.8]). If $|M^n|^*$ is getting large, not much of the martingale part is left anymore.

3. Third key lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$ (see [9, Lemma 4.9]). Here the previous statement is sharpened: even the Emery metric of the above martingale part is small.

4. Fourth key lemma: there exists $\widetilde{X}^n \in \operatorname{conv}(X_n, \ldots)$ such that $\widetilde{M}^n$ converges in the Emery topology (see Lemma 4.10 in [9]). Forward convex combinations then lead to a sequence of semimartingales, still with the same limit of terminal values such that the martingales parts converge in the Emery topology: if $|M_t^n|^*$ stays small, we can conclude by Burkholder-Davis-Gundy inequality, and when it gets large by the previous considerations. It converges to 0 in the Emery topology anyway.

Proposition 1. *Let $\mathcal{X}_1$ satisfy (NUPBR). Let $\widetilde{X}^n = \widetilde{M}^n + \widetilde{A}^n \in \mathcal{X}_1$ be a sequence of special semimartingales, whose terminal values X_1^n converge to a maximal element h_0 in probability such that $\widetilde{M}^n$ converges in the Emery topology. Then $\widetilde{A}^n$ converges in the Emery topology as well.*

Proof. See [9, Lemma 4.11]. This is again a beautiful trading argument, where the still unclear Emery convergence of the finite variation part is proved. Here the concatenation property is used in full strength, in particular we need it for all predictable dynamic trades. □

As already argued above, this proposition together with the key Lemma (iv) implies that $f \in C_0$ yielding that C is in fact weak $*$-closed by step (i) above. Hence the assumptions of the Kreps-Yan Theorem 2 are satisfied and we can conclude (ESM), i.e. the existence of a separating measure.

If the proof was a movie, it would win an Oscar for maintaining tension up to the end, which turns out to be a happy one, since the desired Emery convergence can finally indeed be achieved.

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