Dependence Properties and Comparison Results for Lévy Processes with Applications in Finance

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Outline

1. **Lévy processes and dependence concepts.**

2. **Association of Lévy processes.**

3. **Lévy Copulas.**

4. **Comparison of Lévy processes.**

5. **Applications: Exit times, credit risk, option prices.**

Lévy-Processes

A stochastic process $X = (X_t)$ with values in $I\!R^d$ is called Lévy-process if $X_0 = 0$ and

- X has independent increments,
- X has stationary increments,
- $\forall t, \varepsilon > 0$, $\lim_{h\to 0} P(\|X_{t+h} - X_t\| \geq \varepsilon) = 0$.

The Lévy Measure

Let (X_t) be a Lévy process with values in $\mathbb{R}^d$. The measure ν defined by

$$\nu(A) = E\Big[\big|\{t \in [0,1] \mid \Delta X_t \neq 0,\ \Delta X_t \in A\}\big|\Big]$$

where $A \in \mathbb{B}(\mathbb{R}^d)$ is called the Lévy measure of X.

In what follows we suppose that the Lévy processes under consideration do not have a Brownian part.

The Problem

Characterize the dependence structure of a Lévy process by means of the

(a) Lévy measure.

(b) Lévy copula.

Dependence Concepts

A random vector $X = (X_1, \ldots, X_d)$ is said to be

a) (positively) associated if $Cov(f(X), g(X)) \geq 0$ for all increasing functions $f, g : \mathbb{R}^d \to \mathbb{R}$.

b) positive orthant dependent (POD) if for all $t \in \mathbb{R}^d$

$$F_X(t) \geq \prod_{k=1}^{d} F_{X_k}(t_k) \quad \text{and} \quad \bar{F}_X(t) \geq \prod_{k=1}^{d} \bar{F}_{X_k}(t_k).$$

c) positive supermodular dependent (PSMD) if $Ef(X^\perp) \leq Ef(X)$ for all $f : \mathbb{R}^d \to \mathbb{R}$ supermodular for which the expectation exists where $X^\perp$ is the vector with same marginals as X but independent components.

Note that we have $Association \quad \Rightarrow \quad PSMD \quad \Rightarrow \quad POD.$

Dependence Concepts for Stochastic Processes

The $\mathbb{R}^d$-valued stochastic process $X = (X(t))_{t \geq 0}$ is said to be associated (POD, PSMD) if and only if

$$(X(t_1), \ldots, X(t_n))$$

is associated (POD, PSMD) for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$.

Note that in case of Lévy processes this is equivalent to $X(t)$ being associated (POD, PSMD) for all $t \geq 0$.

Characterization of Dependence by the Lévy Measure

Theorem 1: Let X be an $\mathbb{R}^d$-valued Lévy process with Lévy measure ν.

a) X is associated if and only if ν is concentrated on

$$\mathbb{R}^d_{+,-} = \{x \in \mathbb{R}^d \mid \; x_i \geq 0 \; \forall i \text{ or } x_i \leq 0 \; \forall i\}$$

b) The following statements are equivalent:

(i) X is associated.
(ii) X is POD.
(iii) X is PSMD.

Remark: $X(t)$ associated for a fixed $t > 0$ does not imply that X is associated.

Resnick (1988), Samorodnitsky (1995), Liggett (2005), B., Blatter and Müller (2007)

Lévy Copulas

Let X be an $\mathbb{R}^d$-valued Lévy process. The tail integral of X is the function $U : (\mathbb{R}\backslash\{0\})^d \to \mathbb{R}$ defined by

$$U(x_1, \dots, x_d) := \prod_{i=1}^d \operatorname{sgn}(x_i)\nu\Big(\prod_{j=1}^d \mathcal{I}(x_j)\Big),$$

where $\mathcal{I}(x) = (x, \infty)$ if $x \geq 0$ and $(-\infty, x]$ if $x < 0$. Then there exists a measure defining function $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ with univariate marginals which are the identity functions on $\bar{\mathbb{R}}$ such that

$$U(x_1, \dots, x_d) = F\big(U_1(x_1), \dots, U_d(x_k)\big)$$

for all $x \in \big(\mathbb{R} \setminus \{0\}\big)^d$. This equality has also to be true for all marginals. F is unique on $\prod_{i=1}^d \overline{\operatorname{Ran} U_i}$ and is called Lévy copula of X.

Cont and Tankov (2004), Kallsen and Tankov (2006)

Characterization of Dependence by the Lévy Copula

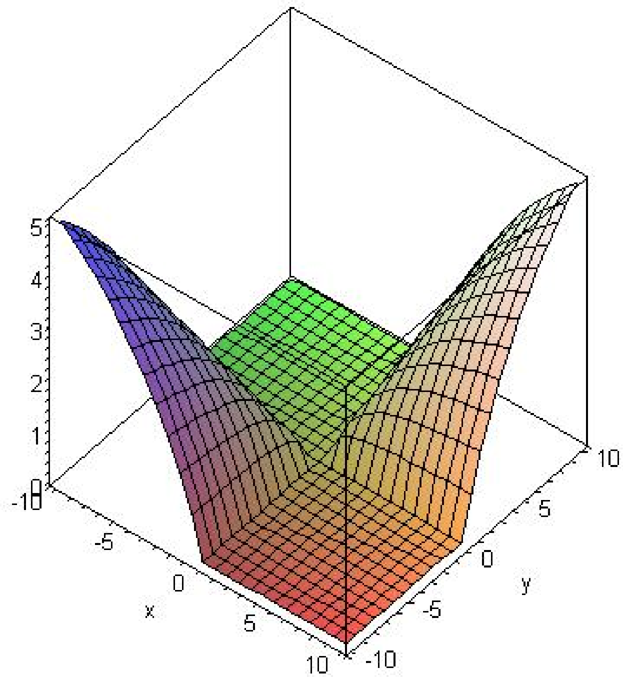
Theorem 2: Let X be an R^d-valued Lévy process with Lévy copula F. X is associated (POD, PSMD) if and only if $F(u) = 0$ for all $u \notin \mathbb{R}^d_{+,-}$ where F is uniquely defined.

Example: (Clayton Lévy Copula)

$$F_\theta(u_1, u_2) = (|u_1|^{-\theta} + |u_2|^{-\theta})^{-\frac{1}{\theta}} 1_{[u \in \mathbb{R}^2_{+,-}]}$$

where $\theta > 0$.

In the figure: $\theta = 1$.



Warning!!

The Lévy copula is not sufficient to characterize all types of dependence. E.g. properties like Conditionally increasing in sequence or MTP_2 cannot be characterized.

Example: X is a 2-dimensional compound Poisson process with

$$\nu = \frac{1}{3}(\delta_{(1,0)} + \delta_{(2,1)} + \delta_{(3,3)}),$$

and Y is a 2-dimensional compound Poisson process with

$$\nu = \frac{1}{3}(\delta_{(1,1)} + \delta_{(2,2)} + \delta_{(3,3)}).$$

Both have the same Lévy copula but the first one is neither CIS nor MTP_2.

Stochastic Orders for Processes

$X = (X_1, ..., X_d)$ is smaller than $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$ in the

a) supermodular order ($X \le_{sm} \tilde{X}$), if $Ef(X) \le Ef(\tilde{X})$ for all supermodular functions $f : I\!R^d \to I\!R$ such that the expectations exist.

b) concordance order, ($X \le_c \tilde{X}$), if both $\bar{F}_X(t) \le \bar{F}_{\tilde{X}}(t)$ and $F_X(t) \le F_{\tilde{X}}(t)$ for all $t \in I\!R^d$ hold.

Two stochastic processes $X = (X(t))_{t \ge 0}$ and $\tilde{X} = (\tilde{X}(t))_{t \ge 0}$ are comparable with respect to the order $\preceq \in \{\le_c, \le_{sm}\}$ ($X \preceq \tilde{X}$) if

$$(X(t_1), \ldots, X(t_n)) \preceq (\tilde{X}(t_1), \ldots, \tilde{X}(t_n))$$

for all $0 \le t_1 < t_2 < \ldots < t_n$ and all $n \in I\!N$.

Comparison Results for Lévy Processes

$\mathcal{B}_0 := \left\{ f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ mb, bounded, } \limsup_{x \to 0} \frac{|f(x)|}{\|x\|^2} < \infty \right\}$

Theorem 3: For Lévy processes X, $\tilde{X}$ with Lévy measures $\nu, \tilde{\nu}$, the following conditions are equivalent:

(i) $X \leq_{sm} \tilde{X}$.
(ii) $\nu \leq_{sm} \tilde{\nu}$, i.e. $\int f d\nu \leq \int f d\tilde{\nu}$ for all supermodular $f \in \mathcal{B}_0$.

Theorem 4: Let $d = 2$. For Lévy processes X, $\tilde{X}$ with Lévy measures ν, $\tilde{\nu}$ and Lévy copulas F, $\tilde{F}$ the following conditions are equivalent:

(i) $X \leq_c \tilde{X}$.
(ii) $X \leq_{sm} \tilde{X}$.
(iii) ν and $\tilde{\nu}$ have the same marginal tail integrals and $F \leq \tilde{F}$.

Bergenthum and Rüschendorf (2006), B., Blatter and Müller (2007)

Applications: Exit Times

Suppose the Lévy process $X = (X_1(t), \ldots, X_d(t))_{t \geq 0}$ represents the evolution of d wealth processes. Denote by

$$\tau_j := \inf\{t \geq 0 \mid X_j(t) \leq c\}$$

the exit time of process $j = 1, \ldots, d$.

Theorem 5: Let X be an $\mathbb{R}^d$-valued Lévy process. If X is associated (or POD or PSMD) then the exit time points $\tau = (\tau_1, \ldots, \tau_d)$ are associated (and thus also POD and PSMD).

Theorem 6: Let X and $\tilde{X}$ be two $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then the exit time points are ordered:

$$\tau = (\tau_1, \ldots, \tau_d) \leq_{sm} \tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_d).$$

Applications: Ruin Times

By $X_t^+ := \sum_{i=1}^d X_i(t)$ we denote the one-dimensional risk process for the insurance company. By ψ_{X^+} we denote its probability of ruin, i.e.

$$\psi_{X^+}(u) = P\left(\inf_{t\geq 0} X_t^+ < 0 \mid X_0^+ = u\right).$$

Theorem 7: Suppose we have two portfolios of risk processes X and $\tilde{X}$ which are both $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then for all $u > 0$:

$$\int_u^\infty \psi_{X^+}(s)ds \leq \int_u^\infty \psi_{\tilde{X}^+}(s)ds.$$

Denuit, Frostig and Levikson (2007), Bregman and Klüppelberg (2006)

Applications: Credit Risk

- Portfolio of d obligors.
- $\lambda_i(t, \Psi_t, Y_t) =$ default intensity of obligor i at time t.
- $\Lambda_i(t) := \int_0^t \lambda_i(s, \Psi_s, Y_s)ds \uparrow \infty$ for $t \to \infty$ a.s.
- $\tau_i := \inf\{t \geq 0 \mid \Lambda_i(t) \geq E_i\}$, default time of obligor i ($E_i \sim \exp(1)$).
- Ψ_t environment process at time t.
- $Y_i(t) = 1_{[E_i,\infty)}(\Lambda_i(t))$, indicator of default.

Applications: Credit Risk

Theorem 8: Assume all necessary measurability and integrability conditions are satisfied and

(i) The environment process (Ψ_t) is associated and has a. s. càdlàg paths.

(ii) The default thresholds E are associated and independent from (Ψ_t).

(iii) For every obligor $i \in \{1, \dots, d\}$, the default intensity $\lambda_i(t, \psi, y)$ is increasing in ψ and y and continuous in ψ.

Then the integrated default intensity process $(\Lambda_t) = \big(\Lambda_1(t), \dots, \Lambda_d(t)\big)$is associated and thus $\tau = (\tau_1, \dots, \tau_d)$ is associated.

B. and Schmock (2007)

Applications: Stock Prices

Let X be an $\mathbb{R}^d$-valued Lévy process and let the price processes of d assets satisfy the following stochastic differential equation

$$\begin{aligned} dS_i(t) &= S_i(t-)\big[\mu_i(t)dt + \sigma_i(t-)dX_i(t)\big] \\ S_i(0) &= 1 \end{aligned}$$

where $\mu_i(t), \sigma_i(t)$ are bounded determinsitic càdlàg functions.

Further we assume for all $i = 1, \ldots, d$

(A) $\sigma_i(t)\big(X_i(t) - X_i(t-)\big) \geq -1$ **for all $t \geq 0$.**

Theorem 9: If the Lévy process X is associated (or POD or PSMD), then the price processes are associated (and thus also POD and PSMD).

Applications: Option Pricing

Take a contingent claim with pay-off $H = h(S_1(T), S_2(T))$. Its price is given by $\pi(H) = B_T^{-1} E_Q[h(S_1(T), S_2(T))]$.

Theorem 10: If h is a supermodular function and S_1 and S_2 are associated under Q, then

$$\pi(H) \geq \pi(H^\perp)$$

where $\pi(H^\perp)$ is the price of the same option with independent price processes.

Typical functions h which are supermodular are
$h(x, y) = \big(\min(x, y) - K\big)^+, h(x, y) = \big(x + y - K\big)^+$.