Optimal Investment with Partial Information

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Standard Problem

Maximize utility of final wealth.

$$\max E^P \left[U \left(X_T \right) \right]$$

Model:

$$\begin{aligned} dS_t &= \alpha S_t dt + S_t \sigma dW_t, \\ dB_t &= r B_t dt \end{aligned}$$

$$\begin{aligned} X_t &= \text{portfolio value at } t \\ u_t &= \text{relative portfolio weight in stock at } t \end{aligned}$$

Wealth dynamics

$$dX_t = X_t \left\{ u_t(\alpha - r) + r \right\} dt + u_t X_t \sigma dW_t$$

Standard approaches:

- Dynamic programming. (Merton etc)
- Martingale methods. (Huang etc)

Standard assumption:

- The volatility σ and the mean rate of return α are **known**.

Standard results:

- Very explicit results.

- Nice mathematics.

Sad facts from real life:

- The volatility σ can be estimated with some precision.

- The mean rate of return α can **not** be estimated **at all**.

Example: If $\sigma = 20\%$ and we want a 95% confidence interval for α, we have to observe S for 1600 years.

Reformulated Problem

- Model α as random variable or random process.
- Take the estimation procedure explicitly into account in the optimization problem.

Extended Standard Problem

Model:

$$dS_t = \alpha(t, Y_t) S_t dt + S_t \sigma(t, Y_t) dW_t,$$

- Y is a "hidden Markov process" which cannot be observed directly.
- We can only observe S.

Previous Studies

- Power or exponential utility.
- Y is a diffusion:
  (Genotte, Brennan, Brendle)
- Y is a finite state Markov chain:
  (Bäuerle–Rieder, Nagai–Runggaldier, Haussmann–Sass).

Technique:

- Filtering theory.
- Use conditional density as extended state.
- Dynamic programming.

Results:

- Very nice explicit results.
- Sometimes a bit messy.
- Separate study for each model.

Object of Present Study

- Study a more general problem
- Avoid DynP (regularity, viscosity solutions etc).
- Investigate the **general** structure.

Related Zariphopoulou Problem

$$\max E^P \left[\frac{1}{\gamma} X_T^\gamma \right]$$

$$\begin{aligned} dS_t &= \alpha(t, Y_t) S_t dt + S_t \sigma_t(t, Y_t) dW_t, \\ dY_t &= \mu(t, Y_t) dt + b(t, Y_t) dW_t. \end{aligned}$$

Note:
Both S and Y are **observable**. Same W driving S and Y. (Zariphopoulou allows for general correlation)

Wealth dynamics

$$dX_t = X_t \left\{ u_t(\alpha_t - r) + r \right\} dt + u_t X_t \sigma dW_t$$

For simplicity we put $r = 0$

$$
\begin{cases}
F_t + \sup_u \left\{ u\alpha x F_x + \frac{1}{2} u^2 \sigma^2 x^2 F_{xx} + \mu F_y + \frac{1}{2} b^2 F_{yy} + u x \sigma b F_{xy} \right\} = 0, \\
F(T, s, y) = \frac{x^\gamma}{\gamma}.
\end{cases}
$$

Ansatz:

$$
F(t, x, y) = \frac{x^\gamma}{\gamma} G(t, y),
$$

PDE:

$$
G_t + \frac{1}{2} b^2 G_{yy} + \left\{ \mu + \frac{\gamma \alpha b}{\sigma(1-\gamma)} \right\} G_y + \frac{\gamma \alpha^2}{2\sigma^2(1-\gamma)} G + \frac{\gamma b^2}{2(1-\gamma)} \cdot \frac{G_y^2}{G} = 0
$$

Non linear! We have a problem!

PDE:

$$G_t + \frac{1}{2}b^2 G_{yy} + \left\{ \mu + \frac{\gamma \alpha b}{\sigma(1-\gamma)} \right\} G_y$$

$$+ \frac{\gamma \alpha^2}{2\sigma^2(1-\gamma)} G + \frac{\gamma b^2}{2(1-\gamma)} \cdot \frac{G_y^2}{G} = 0$$

Clever idea by Zariphopoulou:

$$G(t,y) = H(t,y)^{1-\gamma}$$

$$\begin{aligned} H_t + \left\{ \mu + \frac{\alpha\beta}{\sigma} b \right\} H_y + \frac{1}{2} b^2 H_{yy} + \frac{\beta\alpha^2}{2\sigma^2(1-\gamma)} H &= 0, \\ H(T,y) &= 1. \end{aligned}$$

Linear!
Feynman-Kac representation.

Zariphopoulou Result

- Optimal value function

$$V(t,x,y) = \frac{x^\gamma}{\gamma} H(t,y)^{1-\gamma},$$

- H is given by PDE or by

$$H(t,y) = E^0_{t,y}\left[\exp\left\{\frac{1}{2}\int_t^T \frac{\beta\alpha^2}{(1-\gamma)\sigma^2} dt\right\}\right],$$

where the measure Q^0 has likelihood dynamics of the form

$$dL^0_t = L^0_t\left(\frac{\alpha\beta}{\sigma}\right) dW_t.$$

- The optimal control is given by

$$u^*(t,x,y) = \frac{\alpha}{\sigma^2(1-\gamma)} + \frac{b}{\sigma} \cdot \frac{H_y}{H}.$$

What on earth is going on?

Present Paper

Model: $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t,$$

- α and σ are general $\mathbf{F}$-adapted
- $\mathcal{F}_t^S \subseteq \mathcal{F}_t$
- The short rate is assumed to be zero.

Wealth dynamics:

$$dX_t = u_t \alpha_t X_t dt + u_t X_t \sigma_t dW_t,$$

Problem:

$$\max_u E^P \left[U(X_T) \right]$$

over $\mathbf{F}^S$-adapted portfolios.

Strategy

- Start by analyzing the completely observable case.
- Go on to partially observable model.
- Use filtering results to reduce the problem to the completely observable case.

Completely observable case

Model: $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t,$$

- $\mathcal{F}_t = \mathcal{F}_t^W$
- α and σ are general $\mathbf{F}^W$-adapted

Wealth dynamics:

$$dX_t = u_t \alpha_t X_t dt + u_t X_t \sigma_t dW_t,$$

Problem:

$$\max_u E^P \left[U(X_T) \right]$$

over $\mathbf{F}^W$-adapted portfolios.

Martingale approach

Complete market, so we can separate choice of optimal wealth profile X_T from optimal portfolio choice.

$$\max_{X \in \mathcal{F}_T} \quad E^P\left[U(X)\right]$$

s.t. budget constraint

$$E^Q\left[X\right] = x,$$

Rewrite budget as

$$E^P\left[L_T X\right] = x,$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

Lagrangian relaxation

$$\mathcal{L} = E^P\left[U(X)\right] - \lambda\left(E^P\left[L_T X\right] - x\right),$$

Relaxed problem

$$\max_{X} \int_{\Omega} \left\{ U(X) - \lambda \left(L_T X - x \right) \right\} dP.$$

Separable problem with solution

$$U'(X) = \lambda L_T$$

Optimal wealth:

$$X = F\left(\lambda L_T\right),$$

where

$$F = \left(U'\right)^{-1}$$

The Lagrange multiplier is determined by the budget constraint $E^P\left[L_T X\right] = x$.

Power utility

$$X = F(\lambda L_T), \quad F(y) = y^{-\frac{1}{1-\gamma}},$$

Easy calculation gives us.

Result:

- Optimal wealth is given by

$$X = \frac{x}{H_0} \cdot L_T^{-\frac{1}{1-\gamma}},$$

- H_0 is given by

$$H_0 = E^P\left[L_T^{-\beta}\right], \quad \beta = \frac{\gamma}{1-\gamma}$$

- Optimal expected utility V_0 is given by

$$V_0 = \frac{x^\gamma}{\gamma} H_0^{1-\gamma}.$$

- This is where the fun starts.

$$H_0 = E^P\left[L_T^{-\beta}\right], \quad \beta = \frac{\gamma}{1-\gamma}$$

Recall

$$L_T = \exp\left\{-\int_0^T \frac{\alpha}{\sigma} dW_t - \frac{1}{2}\int_0^T \frac{\alpha^2}{\sigma^2} dt\right\}.$$

Thus

$$L_T^{-\beta} = \exp\left\{\int_0^T \frac{\beta\alpha}{\sigma} dW_t + \frac{1}{2}\int_0^T \frac{\beta\alpha^2}{\sigma^2} dt\right\}.$$

Define the P-**martingale** L^0 by

$$L_t^0 = \exp\left\{\int_0^t \left(\frac{\beta\alpha}{\sigma}\right) dW_s - \frac{1}{2}\int_0^t \left(\frac{\beta\alpha}{\sigma}\right)^2 ds\right\}$$

We can then write

$$L_T^{-\beta} = L_T^0 \exp\left\{\frac{1}{2}\int_0^T \frac{\beta\alpha^2}{(1-\gamma)\sigma^2} dt\right\}.$$

$$H_0 = E^P \left[L_T^0 \exp\left\{ \frac{1}{2} \int_0^T \frac{\beta\alpha^2}{(1-\gamma)\sigma^2} dt \right\} \right],$$

Since L^0 is a martingale, it defines a change of measure

$$L_t^0 = \frac{dQ^0}{dP}, \quad \text{on } \mathcal{F}_t,$$

Thus

$$H_0 = E^0 \left[\exp\left\{ \frac{1}{2} \int_0^T \frac{\beta\alpha^2}{(1-\gamma)\sigma^2} dt \right\} \right],$$

where L^0 has P-dynamics

$$dL_t^0 = L_t^0 \left(\frac{\beta\alpha}{\sigma} \right) dW_t,$$

Results

- Optimal wealth is given by

$$X = \frac{x}{H_0} \cdot L_T^{-\frac{1}{1-\gamma}},$$

- H_0 is given by

$$H_0 = E^0 \left[\exp \left\{ \frac{1}{2} \int_0^T \frac{\beta \alpha_t^2}{(1-\gamma)\sigma_t^2} dt \right\} \right],$$

- $L^0 = dQ^0/dP$ has dynamics

$$dL_t^0 = L_t^0 \left(\frac{\beta \alpha_t}{\sigma_t} \right) dW_t,$$

- Optimal expected utility V_0 is given by

$$V_0 = \frac{x^\gamma}{\gamma} H_0^{1-\gamma}.$$

This can in fact be extended

Results in the observable case

- The optimal wealth process is given by

$$X_t^\star = x\frac{H_t}{H_0} \cdot L_t^{-\frac{1}{1-\gamma}},$$

- H_t is given by

$$H_t = E^0\left[\exp\left\{\frac{1}{2}\int_t^T \frac{\beta\alpha_s^2}{(1-\gamma)\sigma_s^2}ds\right\}\Bigg|\, \mathcal{F}_t\right],$$

- The optimal expected utility process V_t is given by

$$V_t = \frac{(X_t^\star)^\gamma}{\gamma}H_t^{1-\gamma}.$$

- $L^0 = dQ^0/dP$ has dynamics

$$dL_t^0 = L_t^0\left(\frac{\beta\alpha_t}{\sigma_t}\right)dW_t,$$

Furthermore

- The optimal portfolio process is given by

$$u_t^* = \frac{\alpha_t}{\sigma_t^2(1-\gamma)} + \frac{1}{\sigma_t}\frac{\sigma_H}{H}$$

where

$$dH_t = \mu_H dt + \sigma_H dW_t$$

Partially observable case

Model: $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t,$$

- $\mathcal{F}_t^S \subseteq \mathcal{F}_t$
- α is only $\mathbf{F}$-adapted and thus not directly observable.
- σ is $\mathcal{F}_t^S$-adapted (WLOG).

Wealth dynamics:

$$dX_t = u_t \alpha_t X_t dt + u_t X_t \sigma_t dW_t,$$

Problem:

$$\max_u E^P \left[U(X_T) \right]$$

over $\mathbf{F}^S$-adapted portfolios.

Recap on FKK filtering theory

Given some filtration $\mathbf{F}$:

$$\begin{aligned} dY_t &= a_t dt + dM_t \\ dZ_T &= b_t dt + dW_t \end{aligned}$$

Here all processes are $\mathbf{F}$ adapted and

$$\begin{aligned} Y &= \text{signal process,} \\ Z &= \text{observation process,} \\ M &= \text{martingale w.r.t. } \mathbf{F} \\ W &= \text{Wiener w.r.t. } \mathbf{F} \end{aligned}$$

We assume (for the moment) that M and W are **independent**.

Problem:
Compute (recursively) the filter estimate

$$\hat{Y}_t = E\left[Y_t | \mathcal{F}_t^Z \right]$$

The innovations process

Recall $\mathbf{F}$-dynamics of Z

$$dZ_t = b_t dt + dW_t$$

Our best guess of b_t is $\hat{b}_t$, so the genuinely new information should be

$$dZ_t - \hat{b}_t dt$$

The **innovations process** $\bar{W}$ is defined by

$$\bar{W}_t = dZ_t - \hat{b}_t dt$$

Theorem: The process $\bar{W}$ is $\mathbf{F}^Z$-Wiener.

Thus the $\mathbf{F}^Z$-dynamics of Z are

$$dZ_t = \widehat{b}_t dt + d\bar{W}_t$$

Back to the model

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t,$$

Define Z by

$$dZ_t = \frac{1}{S_t \sigma_t} dS_t$$

i.e.

$$dZ_t = \frac{\alpha_t}{\sigma_t} dt + dW_t$$

We then have

$$dZ_t = \frac{\widehat{\alpha}_t}{\sigma_t} dt + d\bar{W}_t$$

where $\bar{W}$ is $\mathbf{F}^S$-Wiener.

Thus we have price dynamics

$$dS_t = \widehat{\alpha}_t S_t dt + S_t \sigma_t d\bar{W}_t,$$

We are back in the completely observable case!

The $mathbfF^S$ martingale measure $\bar{Q}$ is defined by

$$\frac{d\bar{Q}}{dP} = \bar{L}_t, \quad \text{on } \mathcal{F}_t^S, \tag{1}$$

with L given by

$$d\bar{L}_t = \bar{L}_t \left(-\frac{\hat{\alpha}}{\sigma} \right) d\bar{W}_t. \tag{2}$$

The measure $\bar{Q}^0$ is defined by

$$\frac{d\bar{Q}^0}{dP} = \bar{L}_t^0, \quad \text{on } \mathcal{F}_t^S,$$

with $\bar{L}^0$ given by

$$d\bar{L}_t^0 = \bar{L}_t^0 \left(\frac{\hat{\alpha}\beta}{\sigma} \right) d\bar{W}_t.$$

Main results

With notation as above, the following hold.

- The optimal wealth process $\bar{X}^*$ is given by

$$\bar{X}_t^* = x \cdot \frac{\bar{H}_t}{\bar{H}_0} \bar{L}_t^{-\frac{1}{1-\gamma}},$$

where

$$\bar{H}_t = E^{\bar{0}} \left[\exp \left\{ \frac{1}{2} \int_t^T \frac{\beta \hat{\alpha}^2}{(1-\gamma)\sigma^2} ds \right\} \middle| \mathcal{F}_t^S \right],$$

and the expectation is taken under $\bar{Q}^0$.

- The optimal portfolio weight $\bar{u}^*$ is given by

$$\bar{u}^* = \frac{\hat{\alpha}}{\sigma^2(1-\gamma)} + \frac{1}{\sigma} \cdot \frac{\sigma_{\bar{H}}}{\bar{H}},$$

where $\sigma_{\bar{H}}$ is the diffusion term of $\bar{H}$, i.e. $\bar{H}$ has dynamics of the form

$$d\bar{H}_t = \mu_{\bar{H}}(t)dt + \sigma_{\bar{H}}(t)d\bar{W}_t.$$

Results ctd

Furthermore, the optimal utility process $\bar{V}_t$ is given by

$$V_t = \frac{\left(\bar{X}_t^*\right)^\gamma}{\gamma} \bar{H}_t^{1-\gamma},$$

The Markovian Case

Model:

$$
\begin{aligned}
dS_t &= \alpha(t, Y_t) S_t dt + S_t \sigma dW_t, \\
dY_t &= \mu(t, Y_t) dt + b(t, Y_t) dV_t,
\end{aligned}
$$

- For simplicity we assume that W and V are independent Wiener.
- We can observe S but not Y.
- Note that σ cannot depend upon Y.

Our general results still hold, so again we project onto $\mathbf{F}^S$ and obtain

$$
dS_t = \alpha\widehat{(t, Y_t)} S_t dt + S_t \sigma d\bar{W}_t,
$$

We now assume that Y has a conditional density process $p_t(y)$ w.r.t. Lebesgue measure.

Recall

$$dS_t = \widehat{\alpha(t, Y_t)} S_t dt + S_t \sigma d\bar{W}_t,$$

The conditional density p_t satisfies the DMZ equation

$$\begin{aligned} dp_t(y) &= \mathcal{A}^\star p_t(y) dt \\ &+ p_t(y) \left\{ \alpha(t, y) - \int_R \alpha(t, y) p_t(y) dy \right\} d\bar{W}_t \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= \frac{\partial}{\partial t} + \mu(t, y) \frac{\partial}{\partial y} + \frac{1}{2} \sigma^2(t, y) \frac{\partial^2}{\partial y^2} \\ d\bar{W}_t &= \frac{1}{S_t \sigma} \cdot dS_t - \frac{\hat{\alpha}(t, p_t)}{\sigma} dt \\ \hat{\alpha}(t, p) &= \int_R \alpha(t, y) p(y) dy \end{aligned}$$

The pair (S, p) is Markov!

We need to compute things like

$$\bar{H}_t = E^0 \left[\exp\left\{ \frac{1}{2} \int_t^T \frac{\beta \widehat{\alpha}_s^2}{(1-\gamma)\sigma^2} ds \right\} \middle| \mathcal{F}_t^S \right],$$

Now

$$\alpha \widehat{(t, Y_t)} = \widehat{\alpha}(t, p_t),$$

so $\bar{H}_t$ is of the form

$$\bar{H}_t = H(t, p_t)$$

The pair (S, p) is Markov so we can use Kolmogorov.

Result

- The optimal value function V is given by

$$V(t,x,q) = \frac{x^\gamma}{\gamma}\bar{H}(t,q)^{1-\gamma},$$

$$\bar{H}(t,p) = E^0_{t,q}\left[\exp\left\{\frac{1}{2}\int_t^T \frac{\beta\hat{\alpha}^2(s,p_s)}{(1-\gamma)\sigma^2}ds\right\}\right],$$

- The measure $\bar{Q}^0$ has likelihood dynamics

$$d\bar{L}^0_t = \bar{L}^0_t\left(\frac{\hat{\alpha}(t,p_t)\beta}{\sigma}\right)dW_t.$$

- The optimal control is given by

$$u^*(t,q) = \frac{\hat{\alpha}(t,p)}{\sigma^2(1-\gamma)} + \frac{1}{\sigma^2}\cdot\frac{\bar{H}_p(t,p)[\alpha p]}{H(t,p)},$$

- H satisfies an infinite dimensional parabolic PDE.

What on earth is going on?

- What is the economic significance of the Q^0?
- For log utility $Q^0 = P$.
- For exponential utility $Q^0 = Q$.

???

The FKK filter equations

For the model

$$\begin{aligned} dY_t &= a_t dt + dM_t \\ dZ_T &= b_t dt + dW_t \end{aligned}$$

where M and W are independent, we have the FKK non-linear filter equations

$$\begin{aligned} d\widehat{Y}_t &= \widehat{a}_t dt + \left\{ \widehat{Y_t b_t} - \widehat{Y}_t \widehat{b}_t \right\} d\bar{W}_t \\ d\bar{W}_t &= dZ_t - \hat{b}_t dt \end{aligned}$$

Remark: It is easy to see that

$$h_t = E\left[\left(Y_t - \widehat{Y}_t \right) \left(b_t - \widehat{b}_t \right) \middle| \mathcal{F}_t^Z \right]$$