Bounds for Asian basket options

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Mid-Term Conference on Advanced Mathematical Methods for Finance
Vienna, September, 17th-22nd, 2007

Agenda

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[1] Vanmaele M., Deelstra G., Liinev J., Dhaene J. and Goovaerts M. J. (2006). "Bounds for the price of discrete sampled arithmetic Asian options", Journal of Computational and Applied Mathematics, 185(1), 51-90.

[2] Deelstra G., Liinev J. and Vanmaele M. (2004). "Pricing of arithmetic basket options by conditioning", Insurance: Mathematics & Economics, 34, 55-57.

[3] Thompson G.W.P. (1999a). "Fast narrow bounds on the value of Asian options", Working paper, University of Cambridge.

[4] Lord R. (2006). "Partially exact and bounded approximations for arithmetic Asian options", Journal of Computational Finance, Vol 10(2).

1. Introduction: problem description and motivation

- Bounds for European-style discrete arithmetic Asian basket options in a Black & Scholes framework

- Consider a basket with n assets whose prices $S_i(t)$, $i = 1, \ldots, n$, are described, under the risk neutral measure $\mathbb{Q}$ and with r some risk-neutral interest rate, by

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t),$$

- Assume that the different asset prices are instantaneously correlated in a constant way i.e.

$$\text{corr}(dW_i, dW_j) = \rho_{ij}dt.$$

- Given the above dynamics, the i-th asset price at time t equals

$$S_i(t) = S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma_i W_i(t)}$$

- Price of a discrete arithmetic Asian basket call option with a fixed strike K and maturity T on m averaging dates at current time $t = 0$ is determined by

$$ABC(n,m,K,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l\left(T-j\right) - K\right)_+\right]$$

with a_l and b_j positive coefficients, which both sum up to 1.

- For $T \leq m-1$, the Asian basket call option is said to be in progress and for $T > m-1$, we call it forward starting
- Suitable for hedging as their payoff depend on an average of asset prices at different times and of different assets, and takes the correlations between the assets in the basket into account
- No closed-form solutions available in the Black & Scholes setting
- Dahl and Benth (2001a,b)[5] value such options by quasi-Monte Carlo techniques and singular value decomposition
- In the setting of Asian options, Thompson (1999a) used a first order approximation of the arithmetic sum and derived an upper bound that sharpens those of Rogers and Shi (1995)[6]

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[5] Dahl L.O. and Benth F.E. (2001a). "Valuing Asian basket options with quasi-Monte Carlo techniques and singular value decomposition", Pure Mathematics, 2 February, 1-21.

Dahl L.O. and Benth F.E. (2001b). "Fast evaluation of the Asian basket option by singular value decomposition", Pure Mathematics, 8 March, 1-14.

[6] Rogers L.C.G. and Shi Z. (1995). "The value of an Asian option", Journal of Applied Probability, 32, 1077-1088.

- Lord (2006) revised Thompson's method and proposed a shift lognormal approximation to the sums and he included a supplementary parameter which is estimated by an optimization algorithm

- Deelstra et al. (2004) and Vanmaele et al. (2006) used techniques based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas et al. (2000)[7] and Dhaene et al. (2002a)[8], combined with conditioning techniques and splitting up like in Curran (1994)[9], Rogers and Shi (1995) and Nielsen and Sandmann (2003)[10]

- A random vector $(X_1^c, \ldots, X_k^c)$ is *comonotonic* if each two possible outcomes $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of $(X_1^c, \ldots, X_k^c)$ are ordered componentwise.

- Consider $\mathbb{S} = \sum_{i=1}^k X_i$ with $\underline{X} = (X_1, \ldots, X_k)$ with known marginal distributions but unknown dependence structure. Then the sum $\mathbb{S}$ is bounded below and above in convex order $(\preceq_{cx})$ by sums of comonotonic variables:

$$\mathbb{S}^\ell \preceq_{cx} \mathbb{S} \preceq_{cx} \mathbb{S}^u \preceq_{cx} \mathbb{S}^c,$$

[7] Kaas R., Dhaene J. and Goovaerts M.J. (2000). "Upper and lower bounds for sums of random variables", Insurance: Mathematics & Economics, 27, 151-168.

[8] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. and Vyncke D. (2002a). "The concept of comonotonicity in actuarial science and finance: theory", Insurance: Mathematics & Economics, 31(1), 3-33.

[9] Curran M. (1994). "Valuing Asian and portfolio by conditioning on the geometric mean price", Management science, 40, 1705-1711.

[10] Nielsen J.A. and Sandmann K. (2003). "Pricing Bounds on Asian options", Journal of Financial and Quantitative Analysis, 38(2), 449-473.

which implies by definition of convex order that

$$\mathbb{E}\left[\left(\mathbb{S}^{\ell}-d\right)_{+}\right] \leq \mathbb{E}\left[(\mathbb{S}-d)_{+}\right] \leq \mathbb{E}\left[(\mathbb{S}^{u}-d)_{+}\right] \leq \mathbb{E}\left[(\mathbb{S}^{c}-d)_{+}\right]$$

for all d in $\mathbb{R}^{+}$, while $\mathbb{E}\left[\mathbb{S}^{\ell}\right] = \mathbb{E}\left[\mathbb{S}\right] = \mathbb{E}\left[\mathbb{S}^{u}\right] = \mathbb{E}\left[\mathbb{S}^{c}\right]$.

2. Bounds based on conditioning and/or comonotonicity reasoning

- Write the double sum $\mathbb{S} = \sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l\,(T-j)$ as:

$$\mathbb{S} = \sum_{i=1}^{mn} X_i = \sum_{i=1}^{mn} \alpha_i e^{Y_i}$$

with

$$\alpha_i = a_{\lceil \frac{i}{m} \rceil} b_{(i-1) \bmod m} S_{\lceil \frac{i}{m} \rceil}(0) e^{(r-\frac{1}{2}\sigma^2_{\lceil \frac{i}{m} \rceil})(T-(i-1) \bmod m)}$$

and

$$Y_i = \sigma_{\lceil \frac{i}{m} \rceil} W_{\lceil \frac{i}{m} \rceil}(T-(i-1) \bmod m) \backsim \mathcal{N}(0, \sigma^2_{Y_i} = \sigma^2_{\lceil \frac{i}{m} \rceil}(T-(i-1) \bmod m))$$

for all $i = 1, \ldots, mn$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x and

$$y \bmod m = y - \lfloor y/m \rfloor m;$$

$\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

Comonotonic Upper Bound

- Define $\mathbb{S}^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_{nm}}^{-1}(U)$ with U a Uniform$(0,1)$ random variable and consider

$$ABC(n,m,K,T) \leq e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^c - K)_+\right]$$

- Comonotonic upper bound given by

$$ABC(n,m,K,T) \leq \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{-rj}\Phi\left[\sigma_l\sqrt{T-j} - \Phi^{-1}(F_{\mathbb{S}^c}(K))\right] - e^{-rT}K\left(1 - F_{\mathbb{S}^c}(K)\right)$$

where the cdf of the comonotonic sum $F_{\mathbb{S}^c}(K)$ solves

$$\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0)\exp\left[(r - \frac{1}{2}\sigma_l^2)(T-j) + \sigma_l\sqrt{T-j}\,\Phi^{-1}(F_{\mathbb{S}^c}(K))\right] = K$$

where Φ is the cumulative distribution function of a normally distributed random variable.

Interpretation of comonotonic upper bound

The payoff of the Asian basket call option satisfies

$$\left(\sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l(T-j) - K\right)_+ \leq \sum_{l=1}^{n} a_l \left(\sum_{j=0}^{m-1} b_j S_l(T-j) - K_l\right)_+ \leq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \left(S_l(T-j) - K_{lj}\right)_+$$

as well as

$$\left(\sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l(T-j) - K\right)_+ \leq \sum_{j=0}^{m-1} b_j \left(\sum_{l=1}^{n} a_l S_l(T-j) - K_j\right)_+ \leq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \left(S_l(T-j) - K_{lj}\right)_+ ,$$

with

$$\sum_{l=1}^{n} a_l K_l = \sum_{j=0}^{m-1} b_j K_j = \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j K_{lj} = K.$$

By a no-arbitrage argument we find that the time zero price of such Asian basket option should satisfy the following two relations:

$$ABC(n,m,K,T) \leq \sum_{l=1}^{n} a_l AC_l(m, K_l, T) \leq \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j e^{-rj} C_l(K_{lj}, T-j)$$

$$ABC(n,m,K,T) \leq \sum_{j=0}^{m-1} b_j e^{-rj} BC(n, K_j, T-j) \leq \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j e^{-rj} C_l(K_{lj}, T-j).$$

Superreplication by a static portfolio of vanilla call options C_l on the underlying assets S_l in the basket and with different maturities and strikes.

Also an average of Asian options AC_l or a combination of basket options BC with different maturity dates form a superreplicating strategy.

Simon et al. (2000),

Albrecher et al. (2005)

Hobson et al. (2005) for a basket option in a model-free framework
Chen et al. (2007) for a class of exotic options in a model-free framework.

Comonotonic Lower Bound

- A lower bound, in the sense of convex order, for $\mathbb{S} = \sum_{i=1}^{mn} X_i$ is

$$\mathbb{S}^{\ell} = \mathbb{E}\left[\mathbb{S} \mid \Lambda\right]$$

where Λ is a normally distributed random variable such that $(W_l(T-j), \Lambda)$ are bivariate normally distributed for all l and j, which we assume in the sequel.

- Define $r_{l,j}$ by

$$r_{l,j} = \frac{\text{Cov}(W_l(T-j), \Lambda)}{\sigma_\Lambda \sqrt{T-j}}$$

- Assume that all $r_{l,j}$ are positive: then the comonotonic lower bound is determined by (and the case of all $r_{l,j}$ negative can be treated in a similar way):

$$\begin{aligned} ABC(n,m,K,T) \geq & \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{-rj} \Phi\left[r_{l,j}\sigma_l\sqrt{T-j} - \Phi^{-1}(F_{\mathbb{S}^{\ell}}(K))\right] \\ & - e^{-rT} K(1 - F_{\mathbb{S}^{\ell}}(K)) \end{aligned}$$

where $F_{\mathbb{S}^{\ell}}(K)$, solves

$$K = \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) \exp\left[r_{l,j}\sigma_l \Phi^{-1}(F_{\mathbb{S}^{\ell}}(K))\sqrt{T-j} + (r - \frac{1}{2} r_{l,j}^2 \sigma_l^2)(T-j)\right]$$

Choice of the Conditioning random variable

- In order to get a good lower bound, Λ and $\mathbb{S}$ should be as alike as possible
- take $\Lambda = FA1$ or $FA2$ such that for $i = 1, 2$:

$$FAi = \sum_{k=1}^{n} \sum_{p=0}^{m-1} a_k b_p c_i(k,p) \sigma_k S_k(0) W_k(T-p),$$

with

$$c_1(k,p) = e^{\left(r - \frac{1}{2}\sigma_k^2\right)(T-p)}, \qquad c_2(k,p) = 1.$$

or as in Vanduffel et al. (2005)[11] maximizing the first order approximation of the variance of $\mathbb{S}^{\ell}$:

$$FA3 = \sum_{k=1}^{n} \sum_{j=0}^{m-1} a_k b_p S_k(0)\, e^{r(T-j)} \left[(r - \frac{1}{2}\sigma_k^2)\,(T-j) + \sigma_k W_k(T-j) \right]$$

- Nielsen and Sandmann (2003) propose the standardized logarithm GA of the geometric average $\mathbb{G}$ which is defined by

$$\mathbb{G} = \prod_{l=1}^{n} \prod_{j=0}^{m-1} S_l(T-j)^{a_l b_j} = \prod_{l=1}^{n} \left(\prod_{j=0}^{m-1} \left(S_l(0) e^{(r - \frac{1}{2}\sigma_l^2)(T-j) + \sigma_l W_l(T-j)} \right)^{b_j} \right)^{a_l}$$

[11] Vanduffel S., Hoedemakers T. and Dhaene J. (2005). “Comparing Approximations for Risk Measures of Sums of Non-independent Lognormal Random Variables” North American Actuarial Journal, 9(4), 71-82.

Bounds based on the Rogers and Shi approach

- Rogers and Shi approach:

$$ABC(n,m,K,T) \leq e^{-rT}\left\{\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^{\ell}-K)_{+}\right]+\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{\text{Var}(\mathbb{S}\mid\Lambda)}\right]\right\}$$

- Use idea of Nielsen and Sandmann (2003) and use reasoning of Deelstra et al. (2004):

$$ABC(n,m,K,T) \leq e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^{\ell}-K)_{+}\right]+\frac{1}{2}e^{-rT}\int_{-\infty}^{d_\Lambda}(\text{Var}(\mathbb{S}\mid\Lambda=\lambda)^{\frac{1}{2}}f_\Lambda(\lambda)d\lambda$$

where d_Λ is determined such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq K$
and where f_Λ is the normal density function of Λ

- The upper bound UBRSΛ for the Asian basket option price follows:

$$\begin{aligned} ABC(n,m,K,T) \le & e^{-rT} E^{Q}\left[(\mathbb{S}^{\ell}-K)_{+}\right] + \frac{1}{2} e^{-rT} \left\{\Phi\left(d_{\Lambda}^{*}\right)\right\}^{\frac{1}{2}} \\ & \times \left\{ \sum_{l=1}^{n} \sum_{k=1}^{n} \sum_{j=0}^{m-1} \sum_{p=0}^{m-1} a_l a_k b_j b_p S_l(0) S_k(0) \right. \\ & \times e^{r(2T-j-p)} \left(e^{\sigma_l \sigma_k \rho_{lk} \min(T-j,T-p)} - e^{r_{l,j} r_{k,p} \sigma_l \sigma_k \sqrt{(T-j)(T-p)}} \right) \\ & \left. \times \Phi\left(d_{\Lambda}^{*} - r_{l,j}\sigma_l \sqrt{T-j} - r_{k,p}\sigma_k \sqrt{T-p} \right) \right\}^{\frac{1}{2}} \qquad (1) \end{aligned}$$

with $d_{\Lambda}^{*} = \frac{d_{\Lambda} - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}}$.

- Untill now, the lower bound was only known in a nice formula in the comonotonic situation, so also the upper bound UBRSΛ.

Partially Exact/Comonotonic Upper Bound

- For this upper bound a comonotonic situation is not necessary

- Write:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+ \mid \Lambda\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+ \mid \Lambda\right]\mathbf{1}_{\{\Lambda \geq d_\Lambda\}}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+ \mid \Lambda\right]\mathbf{1}_{\{\Lambda < d_\Lambda\}}\right]\end{aligned}$$

- Use

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}-K)_+ \mid \Lambda\right]\mathbf{1}_{\{\Lambda < d_\Lambda\}}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^u-K)_+\right]\mathbf{1}_{\{\Lambda < d_\Lambda\}}\right].$$

where $\mathbb{S}^u$ defined by $\mathbb{S}^u = F^{-1}_{X_1|\Lambda}(U) + F^{-1}_{X_2|\Lambda}(U) + \cdots + F^{-1}_{X_{nm}|\Lambda}(U)$ with U a Uniform$(0,1)$ random variable
and where d_Λ is determined such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq K$

- We get the following expression

$$
\begin{aligned}
&ABC(n,m,K,T) \qquad (2)\\
\leq & e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)}\Phi\left(r_{l,j}\sigma_l\sqrt{T-j}-d_\Lambda^*\right) - e^{-rT}K(1-\Phi(d_\Lambda^*))\\
&+ e^{-rT}\int_0^{\Phi(d_\Lambda^*)}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)-\frac{1}{2}r_{l,j}^2\sigma_l^2(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\Phi^{-1}(v)}\\
&\qquad \times \Phi\left(\sigma_l\sqrt{\left(1-r_{l,j}^2\right)(T-j)}-\Phi^{-1}(F_{\mathbb{S}^u|V=v}(K))\right)dv-\\
&- Ke^{-rT}\left(\Phi(d_\Lambda^*)-\int_0^{\Phi(d_\Lambda^*)}F_{\mathbb{S}^u|V=v}(K)dv\right)
\end{aligned}
$$

where $F_{\mathbb{S}^u|V=v}(K)$ follows from

$$
K=\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}\sigma_l^2)(T-j)+r_{l,j}\sigma_l\Phi^{-1}(v)\sqrt{T-j}+\sigma_l\sqrt{(T-j)(1-r_{l,j}^2)}\Phi^{-1}(F_{\mathbb{S}^u|V=v}(K))}. \qquad (3)
$$

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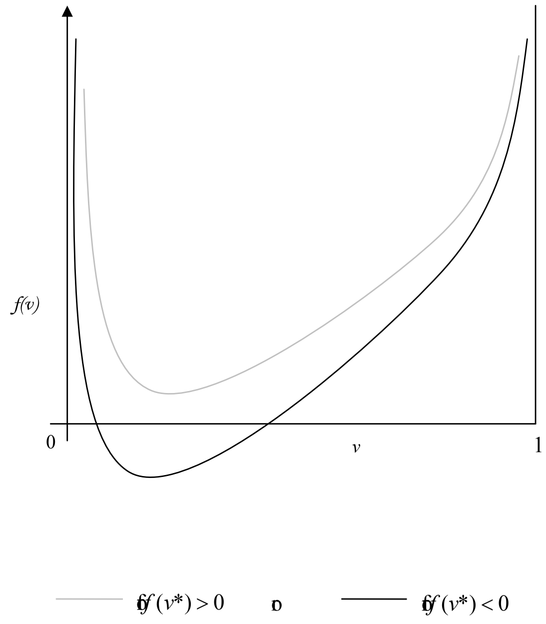
3. Lower bound and upper bound UBRS in the Non-Comonotonic Case

- Not all $r_{l,j}$ have the same sign
- Write like suggested by Lord (2006) in the basket option case:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^{\ell}-K)_{+}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{S}\mid\Lambda\right]-K\right)_{+}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{l=1}^{n}\sum_{j=0}^{m-1}a_l b_j S_l(0)e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\frac{\Lambda-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}}-K\right]_{+} \\ &= \int_0^1\left[\sum_{l=1}^{n}\sum_{j=0}^{m-1}a_l b_j S_l(0)e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\Phi^{-1}(v)}-K\right]_{+}dv\end{aligned}$$

- Denote $f(v)=\sum_{l=1}^{n}\sum_{j=0}^{m-1}a_l b_j S_l(0)e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\Phi^{-1}(v)}-K$.
- Denote v^* the minimum of the function f.


f(v)
0
v
1
f(v*) > 0
f(v*) < 0

- Case 1 of $f(v) \geq 0$ for all v:

$$\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^{\ell} - K)_{+}\right] = \mathbb{E}^{\mathbb{Q}}[\mathbb{S}] - K = \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} - K$$

- Case 2 of $f(v^*) < 0$:

Denote d_{Λ_1} and d_{Λ_2} the two solutions of the following equation in x:

$$\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\frac{x-\mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_\Lambda}} = K$$

with $d_{\Lambda_1} \leq d_{\Lambda_2}$.

Then:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[(\mathbb{S}^{\ell}-K)_{+}\right] \\
&\quad = \int_{-\infty}^{d_{\Lambda_1}} \left(\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\Phi^{-1}(v)} - K \right) f_{\Lambda}(\lambda)d\lambda + \\
&\qquad + \int_{d_{\Lambda_2}}^{\infty} \left(\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}r_{l,j}^2\sigma_l^2)(T-j)+r_{l,j}\sigma_l\sqrt{T-j}\Phi^{-1}(v)} - K \right) f_{\Lambda}(\lambda)d\lambda \\
&\quad = \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} \Phi\left(d_{\Lambda_1}^{*} - r_{l,j}\sigma_l\sqrt{T-j}\right) - K\Phi\left(d_{\Lambda_1}^{*}\right) + \\
&\qquad + \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} \Phi\left(r_{l,j}\sigma_l\sqrt{T-j} - d_{\Lambda_2}^{*}\right) - K\Phi\left(-d_{\Lambda_2}^{*}\right)
\end{aligned}$$

where for $i = 1, 2$ $d_{\Lambda_i}^{*} = \frac{d_{\Lambda_i} - \mathbb{E}^{\mathbb{Q}}[\Lambda]}{\sigma_{\Lambda}}$.

- In our numerical examples, the first integral appears to be negligeable.
- This lower bound can be used in the Rogers & Shi approach.
  So UBRS is possible in a non-comonotonic situation.

4. Generalization of the upper bound of Thompson

- Let

$$f_l(T-j) = \mu_l(T-j) + \overline{\sigma}\left[\sigma_l W_l(T-j) - \sum_{i=1}^{n}\sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T-k)\right]$$

be a random function where $\mu_l(T-j)$ and $\overline{\sigma}$ are deterministic functions with

$$\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \mu_l(T-j) = 1. \tag{4}$$

- Price of an Asian basket option is bounded by

$$ABC(n,m,K,T) \leq e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \mathbb{E}^{\mathbb{Q}}\left[\left(S_l\left(T-j\right) - K f_l(T-j)\ \right)_+\right] \tag{5}$$

- Thompson took $\overline{\sigma} = 1$, but we will optimize over $\overline{\sigma}$

- Consider the Lagrangian:

$$L(\lambda, \{\mu_l(T-j)\}, \overline{\sigma}) = \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \mathbb{E}^{\mathbb{Q}}\left[(S_l(T-j) - K f_l(T-j))_+\right] - \lambda\left(\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \mu_l(T-j) - 1\right)$$

- Consider the first order derivative with respect to $\{\mu_l(T-j)\}$. Equating to zero leads to:

$$\mathbb{Q}\left[Y_l(T-j) \geq K\mu_l(T-j)\right] = -\frac{\lambda}{K}$$

with λ a constant and where

$$Y_l(T-j) = S_l(T-j) - \overline{\sigma}K\left[\sigma_l W_l(T-j) - \sum_{i=1}^{n}\sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T-k)\right],$$

- Thompson suggest to approximate $\exp(\sigma_l W_l(T-j)) \approx 1 + \sigma_l W_l(T-j)$ which is valid for small $\sigma_l\sqrt{T-j}$

and to use as first order approximations for $S_l\,(T-j)$ and $Y_l(T-j)$ respectively:

$$S_l^{FA}\,(T-j) = S_l(0)e^{(r-\frac{1}{2}\sigma_l^2)(T-j)}\,(1+\sigma_l W_l(T-j))$$

$$Y_l^{FA}(T-j) = S_l^{FA}\,(T-j) - \overline{\sigma}K\left[\sigma_l W_l(T-j) - \sum_{i=1}^{n}\sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T-k)\right].$$

- Denote $\mu_l^{FA}(T-j)$ by analogy to $\mu_l(T-j)$ and conclude that

$$\mathbb{Q}\left[Y_l^{FA}(T-j) \geq K\mu_l^{FA}(T-j)\right] = -\frac{\lambda}{K}$$

Using the fact that $Y_l^{FA}(T-j)$ is normally distributed, we deduce that

$$\frac{K\mu_l^{FA}(T-j) - S_l(0)e^{(r-\frac{1}{2}\sigma_l^2)(T-j)}}{\sqrt{\text{Var}\left(Y_l^{FA}(T-j)\right)}} = \gamma^{FA} = \Phi^{-1}(1+\frac{\lambda}{K})$$

and hence

$$\mu_l^{FA}(T-j) = \frac{1}{K}\left(S_l(0)e^{(r-\frac{1}{2}\sigma_l^2)(T-j)} + \gamma^{FA}\sqrt{\text{Var}\left(Y_l^{FA}(T-j)\right)}\right)$$

The constraint (4) for $\mu_l^{FA}(T-j)$ implies that

$$\gamma^{FA} = \frac{K - \sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j S_l(0)e^{(r-\frac{1}{2}\sigma_l^2)(T-j)}}{\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \sqrt{\text{Var}\left(Y_l^{FA}\,(T-j)\right)}}.$$

As the upper bound in (5) holds for any function $\mu_l(T-j)$ satisfying the above contraint, it also holds for the approximately optimal function $\mu_l^{FA}(T-j)$:

$$\begin{aligned}
&ABC(n,m,K,T)\\
&\leq e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1}a_l b_j \mathbb{E}^{\mathbb{Q}}\left[\left(S_l\,(T-j)-\overline{\sigma}K\left(\frac{\mu_l^{FA}(T-j)}{\overline{\sigma}}+\right.\right.\right.\\
&\qquad\left.\left.\left.+\sigma_l W_l(T-j)-\sum_{i=1}^{n}\sum_{k=0}^{m-1}a_i b_k \sigma_i W_i(T-k)\right)\right)_+\right].
\end{aligned}$$

It is well-known that

$$S_l\,(T-j)-\overline{\sigma}K\left(\frac{\mu_l^{FA}(T-j)}{\overline{\sigma}}+\sigma_l W_l(T-j)-\sum_{i=1}^{n}\sum_{k=0}^{m-1}a_i b_k \sigma_i W_i(T-k)\right)$$

conditioned on $W_l(T-j)=x\sqrt{T-j}$ is normally distributed.
Define $c_l^{FA}(T-j,x,\overline{\sigma})$ and $d_l^2(T-j,\overline{\sigma})$ as the corresponding conditional mean and variance.

- The upper bound then becomes with $\phi(\cdot)$ denoting the density of the standard normal distribution and Z having a $\mathcal{N}(0,1)$ distribution:

$$
\begin{aligned}
&ABC(n,m,K,T) \\
&\leq e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{Q}}\left[\left(c_l^{FA}(T-j,x,\overline{\sigma}) + d_l(T-j,\overline{\sigma})Z\right)_+\right]\phi(x)dx \\
&\leq e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty} \left\{ c_l^{FA}(T-j,x,\overline{\sigma})\Phi\left(\frac{c_l^{FA}(T-j,x,\overline{\sigma})}{d_l(T-j,\overline{\sigma})}\right)\right. \\
&\qquad\qquad \left. + d_l(T-j,\overline{\sigma})\phi\left(\frac{c_l^{FA}(T-j,x,\overline{\sigma})}{d_l(T-j,\overline{\sigma})}\right)\right\}\phi(x)dx
\end{aligned}
$$

- Use the algorithm suggested by Lord(2006):
  1. Calculate the upper bound using $\mu_l^{FA}(T-j)$ for three carefully chosen values of $\overline{\sigma}$
  2. Fit a quadratic function in $\overline{\sigma}$ to these computed values
  3. Determine the value of $\overline{\sigma}$ in which the upper bound attains its minimum
  4. Recalculate the upper bound in the approximately optimal $\overline{\sigma}$

5. Generalization of an upper bound of Lord

- Approximate $Y_l(T-j)$ by a shifted lognormal random variable in the form:

$$Y_l^{SLN}(T-j) = \alpha(l,T-j) + \exp\left[\theta(l,T-j) + \omega(l,T-j)Z_l\right]$$

where $\alpha(l,T-j)$, $\theta(l,T-j)$ and $\omega(l,T-j)$ are the shift, mean and volatility functions and Z_l is a standard normal distribution

- Determine the parameters by equating the first three moments of the shifted lognormal to those of the original random variable $Y_l(T-j)$

- Use the same reasoning as before and conclude that

$$\mathbb{Q}\left[Y_l^{SLN}(T-j) \geq K\mu_l^{SLN}(T-j)\right] = -\frac{\lambda}{K}$$

- Replace $\mu_l^{FA}(T-j)$ by $\mu_l^{SLN}(T-j)$

- Upper bound given by

$$ABC(n,m,K,T) \leq e^{-rT}\sum_{l=1}^{n}\sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty}\left\{c_l^{SLN}(T-j,x,\overline{\sigma})\Phi\left(\frac{c_l^{SLN}(T-j,x,\overline{\sigma})}{d_l(T-j,\overline{\sigma})}\right)\right.$$
$$\left.+d_l(T-j,\overline{\sigma})\phi\left(\frac{c_l^{SLN}(T-j,x,\overline{\sigma})}{d_l(T-j,\overline{\sigma})}\right)\right\}\phi(x)dx$$

- Apply minimization algorithm as described before.

6. Numerical results

Notations:

- Λ can be $FA1$, $FA2$, $FA3$ or GA,
- LBΛ for both the comonotonic lower bound and the non-comonotonic lower bound
- PECUBΛ for partially exact/comonotonic upper bound,
- UBRSΛ for upper bound based on the Rogers & Shi approach,
- CUB for comonotonic upper bound
- PECUB $=$ min(PECUB$FA1$, PECUB$FA2$, PECUB$FA3$, PECUBGA),
- UBRS $=$ min(UBRS$FA1$, UBRS$FA2$, UBRS$FA3$, UBRSGA),
- LB $=$ max(LB$FA1$, LB$FA2$, LB$FA3$, LBGA),
- ThompUB for Thompson's upper bound with $\overline{\sigma} = 1$,
- ThompUBquad, and SLNquad for upper bound based on the first order approximation and on the shift lognormal approximations, which use a numerical optimization algorithm to approximate the optimal scale $\overline{\sigma}$.

- The moneyness of the option is defined as

$$\frac{K}{\sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \mathbb{E}^{\mathbb{Q}} \left[S_l(T-j) \right]} - 1.$$

Asian basket option

- Data from Beisser (2001)[12]: Asian basket options with monthly averaging written on a fictitious chemistry-pharma basket that consists of the five German DAX stocks listed in the tables below
- The annual risk-free interest rate r is equal to 6%
- The averaging period of all options is five months and starts five month before maturity

- $FA1$, $FA2$ and $FA3$ lead not to comonotonic lower bounds since the correlations $r_{l,j}$ do not have the same sign

[12] Beisser J. (2001). Topics in Finance - A conditional expectation approach to value, Basket and Spread Options. Ph.D. Thesis, Johannes Gutenberg University Mainz.

stock	initial stock price	weight (in %)	volatility (in %)	dividend yield (in %)
BASF	42.55	25	33.34	2.59
Bayer	48.21	20	31.13	2.63
Degussa-Hüls	34.30	30	33.27	3.32
FMC	100.00	10	35.12	0.69
Schering	66.19	15	36.36	1.24

Table 1: Stock characteristics

	BASF	Bayer	Degussa-Hüls	FMC	Schering
BASF	1.00	0.84	−0.07	0.45	0.43
Bayer	0.84	1.00	0.08	0.62	0.57
Degussa-Hüls	−0.07	0.08	1.00	−0.54	−0.59
FMC	0.45	0.62	−0.54	1.00	0.86
Schering	0.43	0.57	−0.59	0.86	1.00

Table 2: Correlation structure

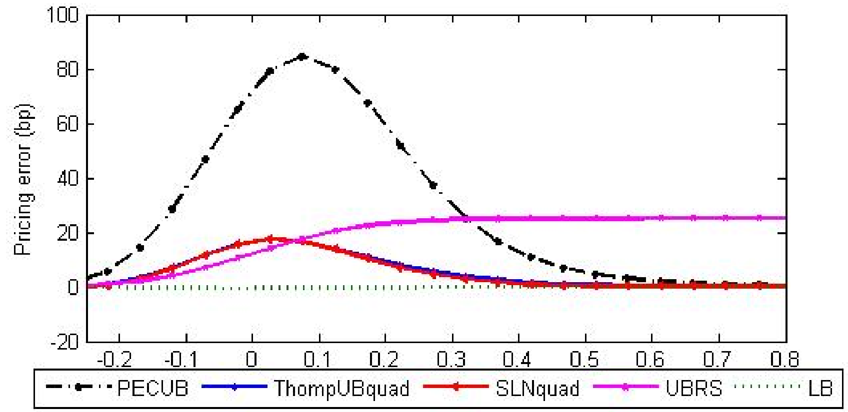


Figure 2: Comparison of option values with $T = 0.5$

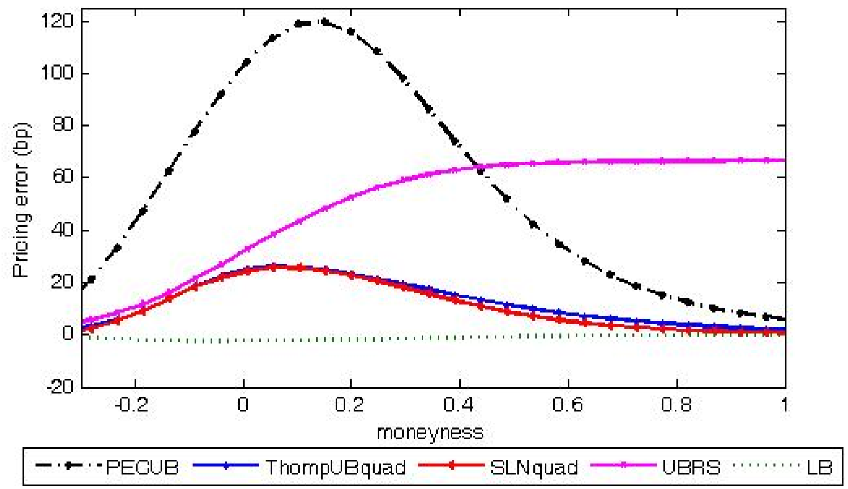


Figure 3: Comparison of option values with $T = 1$

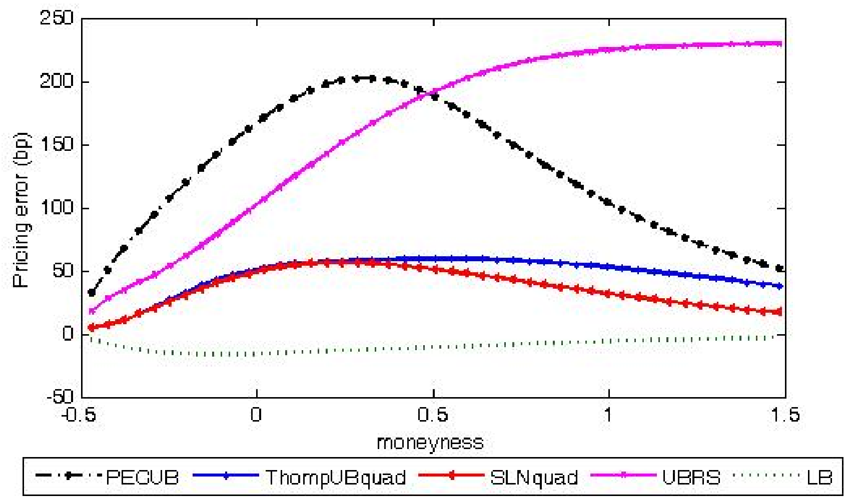


Figure 4: Comparison of option values with $T = 5$

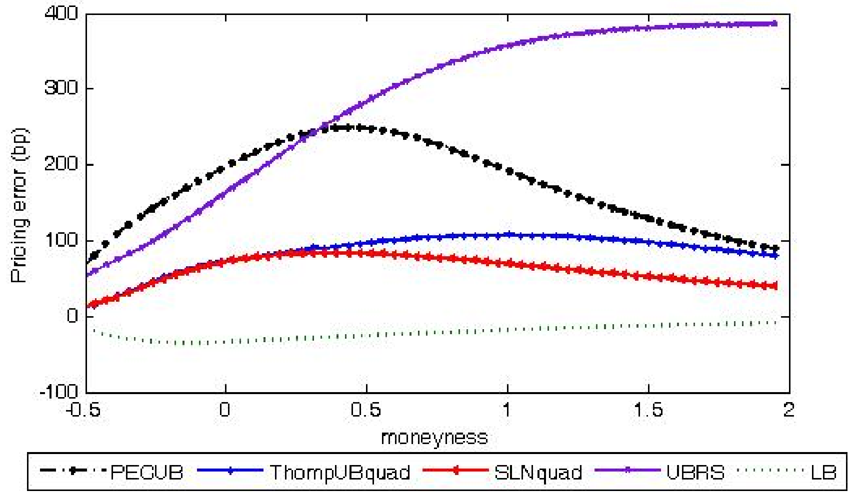


Figure 5: Comparison of option values with $T = 10$

Table 3: Valuation results for Asian basket call option

	K		Thomp	Thomp	SLN				
T	(moneyness)	To compare	UB	UBquad	quad	LBΛ	UBRSΛ	PECUBΛ	Λ
$\frac{1}{2}$	40	MC:10.8465	10.8582	10.8580	10.8520	10.8448	10.8556	10.9042	$FA1$
	(-0.2181)	SE:0.0057				10.8448	10.8558	10.9054	$FA2$
		CUB:11.1221				10.8448	10.8554	10.9023	$FA3$
						10.8414	10.8770	10.9290	GA
	50	MC:2.7860	2.9564	2.9442	2.9415	2.7801	2.8937	3.4912	$FA1$
	(-0.0227)	SE:0.0040				2.7801	2.8930	3.4919	$FA2$
		CUB:4.3465				2.7800	2.8903	3.4376	$FA3$
						2.6705	3.2836	3.9378	GA
	60	MC:0.2338	0.3591	0.3417	0.3361	0.2299	0.4617	0.9288	$FA1$
	(0.1728)	SE:0.0012				0.2299	0.4613	0.9272	$FA2$
		CUB:1.1856				0.2300	0.4573	0.9080	$FA3$
						0.1742	1.1034	1.0407	GA

Table 4: Valuation results for Asian basket call option

T	K (moneyness)	To compare	Thomp UB	Thomp UBquad	SLN quad	LBΛ	UBRSΛ	PECUBΛ	Λ
5	40	MC:17.3030	17.6536	17.6361	17.6159	16.9863	18.1608	18.5893	$FA1$
	(−0.3346)	SE:0.1319				17.0030	18.1698	18.6527	$FA2$
		CUB: 20.2517				16.9727	18.1300	18.5112	$FA3$
						16.9010	18.5126	18.8484	GA
	50	MC:12.5916	13.2374	13.1674	13.1334	12.2352	13.9249	14.5541	$FA1$
	(−0.1807)	SE:0.0295				12.2421	13.8929	14.5678	$FA2$
		CUB:16.4350				12.2282	13.7679	14.1973	$FA3$
						11.9023	14.6519	14.9816	GA
	60	MC:9.1299	10.0549	9.8545	9.8331	8.7834	11.0153	11.6879	$FA1$
	(−0.0168)	SE:0.0268				8.7774	10.9485	11.6439	$FA2$
		CUB:13.4094				8.7853	10.7215	11.0728	$FA3$
						8.2379	12.0024	12.0553	GA
	70	MC: 6.6520	7.8539	7.4618	7.4451	6.3285	9.0801	9.5618	$FA1$
	(0.1470)	SE: 0.0241				6.3127	9.0051	9.5048	$FA2$
		CUB: 11.0082				6.3376	8.6661	8.8910	$FA3$
						5.6654	10.2258	9.7925	GA

Results

- Non-comonotonic lower bounds LB$FA1$, LB$FA2$ and LB$FA3$ perform better than the comonotonic lower bound LBGA
- For short maturities and in- and at-the-money, UBRS outperforms all the other upper bounds
- Other cases: ThompUBquad and SLNquad provide the best upper bounds
- Lower bound is very close to the Monte Carlo value but looses a bit of its sharpeness for larger maturities
- Precision of ThompUBquad and SLNquad decreases with the maturity T
- PECUB is only usefull far-out-of-the-money, otherwise too high to be usefull in comparison with ThompUBquad and SLNquad

7. Conclusion

- Bounds for the price of a discrete arithmetic Asian basket call option with fixed strike and this in the Black and Scholes setting.
- Very good lower bound in all cases: not only in the comonotonic case but also in the non-comonotonic case, which can then be applied to obtain the UBRS upper bound.
- UBRS is the best upper bound for short maturities and in- and at-the-money.
- SLNquad is the best upper bound for all other cases.

THANK YOU FOR YOUR ATTENTION!!!