DYNAMIC RISK MEASURES AND LAW INVARIANCE

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A **dynamic convex (coherent) risk** measure is a map $\rho : \mathcal{R}^\infty_{1,T} \to \mathcal{R}^\infty_{0,T}$, defined on the **process of cash-flows**.

Conventionally, dynamic convex (coherent) risk measures are identified with **dynamic concave (coherent) monetary utilities** as

$$u_t(X) = -\rho_t(X), \qquad X \in \mathcal{R}^\infty_{1,T} \quad \text{(bounded adapted processes, representing cash-flows).}$$

These mappings by definition satisfy

$$\left.\begin{array}{l} \textbf{1. Normalization:} \\ u_t(X\mathbf{1}_{\{s\le t\}}) = 0 \quad \forall X \in \mathcal{R}^\infty_{1,T} \\ \textbf{2. Dynamic Translation invariance:} \\ u_t(X + m\mathbf{1}_{\{t+1\}}) = u_t(X) + m \quad \forall m \in L^\infty(\mathcal{F}_t) \end{array}\right\} \text{dependence only on future}$$

3. Monotonicity: $u_t(X) \le u_t(Y) \quad \forall X \le Y \in \mathcal{R}^\infty_{1,T}$

4. Conditional Concavity:

$u_t\big(\lambda X + (1-\lambda)Y\big) \ge \lambda u_t(X) + (1-\lambda)u_t(Y) \quad \forall \mathcal{F}_t$-measurable $\lambda \in [0,1]$

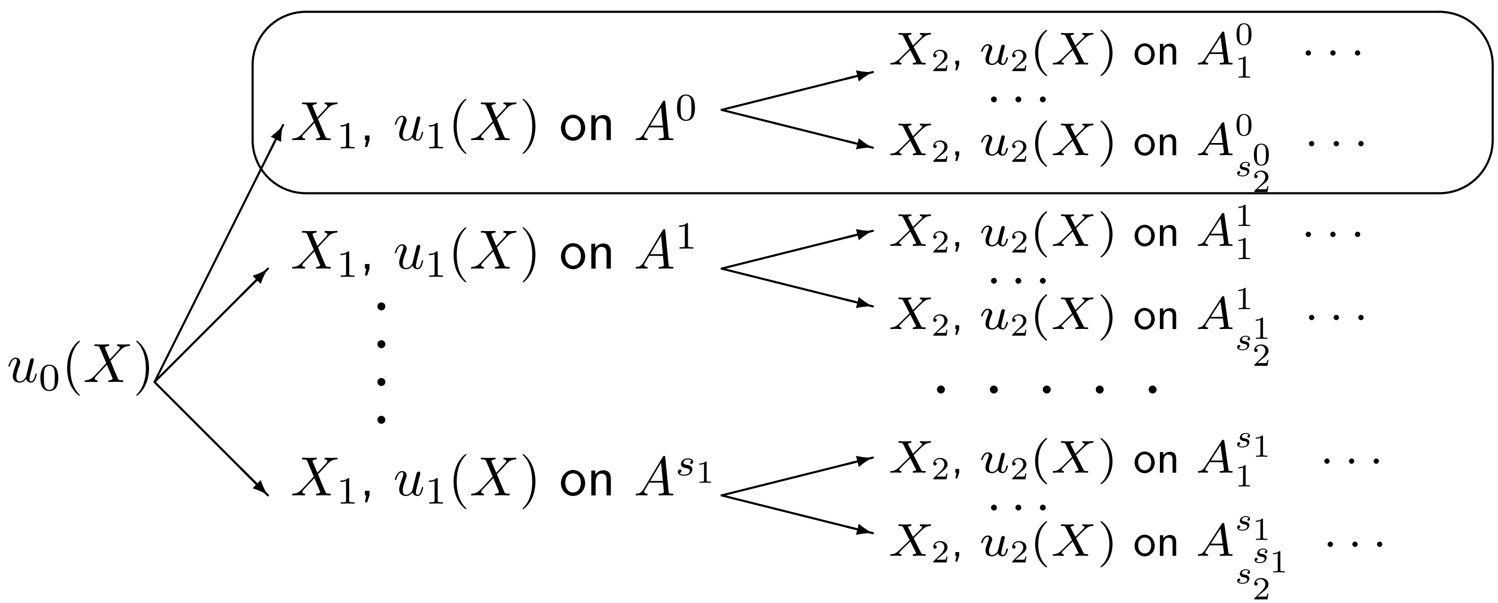
For **coherent** dynamic risk measures (monetary utilities) **conditional positive homogeneity** is additionally assumed:

$$u_t(aX) = au_t(X) \quad \forall X \in \mathcal{R}^\infty_{1,T},\ a \in L^\infty(\mathcal{F}_t),\ a \geq 0.$$

The notion of dynamic risk measure extends naturally the static or rather *one time step* model, initiated in the seminal work of Artzner et. al.: *if* $T = 1$, *then* ρ_0 *is the classical coherent (convex) risk measure.*

Monotonicity and dynamic translation property imply a very natural **local property**:

$$\mathbf{1}_A u_t(X) = \mathbf{1}_A u_t(X\mathbf{1}_A) \quad \forall A \in \mathcal{F}_t.$$



However, to make the dynamic risk measures *"truly dynamical"* in addition we need to assume a kind of *dynamical consistency*. It appears that the property, called (strong) **time-consistency** is quite natural:

$$X_{t+1} = Y_{t+1} \text{ and } u_{t+1}(X) \le u_{t+1}(Y) \quad \Rightarrow \quad u_t(X) \le u_t(Y).$$

In words: *if at time $t+1$ the cash-flows X_{t+1} and Y_{t+1} coincide and we certainly know that on the next date $t+1$ we estimate the future of Y better than X, than at time t we should also estimate Y better than X.*

Time-consistency is equivalent (under the other assumptions) to the **dynamic programming principle**:

$$u_t(X) = u_t\big((X_{t+1} + u_{t+1}(X))\mathbf{1}_{\{t+1\}}\big). \qquad (DPP)$$

This means that to define a time-consistent risk measure we need T 1-step conditional risk measures $L^\infty(\mathcal{F}_t) \to L^\infty(\mathcal{F}_{t-1})$ and then use (DPP).

Law-invariance for the classical static (or one-step) model looks like quite a natural property.

Although it is easy to construct a non-law-invariant convex (coherent) risk measure. But such examples are rather just math phenomena, or *they involve something like dynamic trading in between of the present and final dates and so such approaches change the model for one-step ahead risk.*

Law-invariant static risk measures were characterized by Kusuoka in the coherent case. Kusuoka's formula was extended to the case of convex risk measures by Kunze and independently by Frittelli and Rosazza Gianin.

Roughly this result represents any convex measure of risk as a functional of *"elementary coherent risk measures"*:

$$\text{Tail Value-at-Risk:} \quad \mathrm{TVaR}_\lambda(X) := -\inf\left\{\mathsf{E}_{\mathsf{Q}} X : \frac{d\mathsf{Q}}{d\mathsf{P}} \leq \frac{1}{\lambda}\right\}.$$

For dynamic risk measures a naïve law-invariance is NOT natural under much loose assumptions than was made here.

But what is the "naïve law-invariance"?
I cannot give a precise description but formulate some simple special cases of "naïve law-invariance":

Let the cash flow processes X and Y be non-zero only for the final date T:

$$X = (0, \ldots, 0, X_T), \quad Y = (0, \ldots, 0, Y_T)$$

and $Law(X_T) = Law(Y_T)$. Then it is **not** *natural to think that*

$$u_0(X) = u_0(Y),$$

because at time 0 we might know, that at time $t \in (0, T)$ the conditional laws of X_T and Y_T are not going to be equal: $Law(X_T|\mathcal{F}_t) \neq Law(Y_T|\mathcal{F}_t)$.

Let's develop a simple semi-quantitative example with "naïve law-invariance".

Consider a 2-step model ($T = 2$) and let $u : \mathcal{R}^\infty_{1,2} \to \mathcal{R}^\infty_{0,2}$ be a dynamic coherent risk monetary utility. Assume the following "naïve law-invariance":

1. $u_0(X) = u_0(Y)$ whenever the final capital $X_1 + X_2$ is equal in law to $Y_1 + Y_2$.

2. $u_0(X) = u_1(X)$ if $X_1 = 0$ and Law$(X_2|\mathcal{F}_1) =$ Law(X_2), i.e. X_2 is independent of $\mathcal{F}_1$ — *natural because no new information about the law of X adds at $t = 1$.*

Then u cannot be time-consistent (assuming $(\mathcal{F}_1, \mathcal{F}_2)$ reasonably reach).

Almost a proof: Consider $X = (\xi_1, C\xi_2)$, $C > 0$, and $Y = (0, \sqrt{1 + C^2}\xi_2)$, where $\xi_1, \xi_2 \sim N(0, 1)$ and ξ_2 be independent of $\mathcal{F}_1$. Then the 2nd property imply that $u_1(Y) = \sqrt{1 + C^2}u_1(\xi_2 \mathbf{1}_{t=2})$ is a constant. Therefore, if u were time-consistent by the dynamic programming principle

$$u_0(Y) = u_0\big((0 + \sqrt{1 + C^2}u_1(\xi_2 \mathbf{1}_{t=2}))\mathbf{1}_{t=1}\big) = \sqrt{1 + C^2}u_1(\xi_2 \mathbf{1}_{t=2}).$$

And similarly, $u_0(X) = u_0(\xi_1 \mathbf{1}_{t=1}) + u_1(\xi_2 \mathbf{1}_{t=2})C$. Clearly, $u_0(X) \neq u_0(Y)$ since the one of them is linear in C and the other isn't. But Law$(X_1+X_2) =$ Law(Y_1+Y_2).

However, there are dynamic time-consistent risk measures which appear to be good candidates for "elementary building blocks" for "good" dynamic risk measures.
For a **predictable** $\lambda = (\lambda_1, \ldots, \lambda_T)$, $\lambda_t \in [0, 1]$ a.s., we define **Dynamic TVaR** by the dynamic programming principle. We set $\mathrm{DTVaR}_T^\lambda(X) := 0$ and

$$\mathrm{DTVaR}_t^\lambda(X)(\omega) := \widetilde{\mathrm{TVaR}}^{\lambda_t(\omega)}\Big[\mathsf{Law}\Big(X_{t+1} - \mathrm{DTVaR}_{t+1}^\lambda(X)\Big|\mathcal{F}_t\Big)(\omega)\Big],$$

where $\widetilde{\mathrm{TVaR}}^\alpha$ is a functional on probability distributions, implied by TVaR^α:

$$\widetilde{\mathrm{TVaR}}^\alpha[\mathsf{Law}(\xi)] = \mathrm{TVaR}^\alpha(\xi).$$

The Dynamic TVaR are clearly dynamic time-consistent coherent risk measures.

Unlike the static case here we have a lot of additional freedom in choosing $\lambda = (\lambda_1, \ldots, \lambda_T)$, $\lambda_t \in [0, 1]$.
Is it OK to have them not constant $\lambda_1 = \cdots = \lambda_T$? or allow λ to be any predictable process?
In what follows I'll try to convince that we should exclude the case of random λ_t's.

Let us test a "naïve law-invariance" on DTVaR^{λ} with $\lambda_0 = \cdots = \lambda_T$.
Let $T = 3$ and ξ_1, ξ_2, ξ_3 are i.i.d. Bernoulli variables, taking ± 1 values with $p = \frac{1}{2}$.

	$t=1$	$t=2$	$t=3$
$X_t :=$	0	0	$\begin{cases} 1, & \text{if } \xi_1+\xi_2+\xi_3 > 0, \\ -1, & \text{if } \xi_1+\xi_2+\xi_3 < 0 \end{cases}$
$Y_t :=$	0	0	ξ_3

This is a *Delbaen's* idea: take two dynamic games, both with a cash-flow only on the finite date. In the first one we toss a coin every day and we win/loose, if the number of heads is greater/less than that of tails. In the second we toss a coin just once on the last day.

But $\frac{13}{27} = \mathrm{DTVaR}_0^{(\frac{3}{4},\frac{3}{4},\frac{3}{4})}(X) \gneqq \mathrm{DTVaR}_0^{(\frac{3}{4},\frac{3}{4},\frac{3}{4})}(Y) = \frac{1}{3}$!

In spite of Law$(X_T) =$ Law(Y_T) (because Law$(X_T|\mathcal{F}_1) \neq$ Law$(Y_T|\mathcal{F}_1)$).

It's really amazing that the more predictable X is more risky than the less predictable Y!

Let's discuss this now and forget about the law-invariance for 1 slide.

Already in the static case the coherent risk measures contradict the old out-of-date slogan:

Risk is a measure of uncertainty

— wrong for coherent risk because of cash invariance.

In the dynamic case **time-consistency** contradicts the same re-formulated slogan:

More predictable contingent claim with the same mean means less risky

— wrong for time-consistent risk.

Why? Mathematical reasoning is clear after the first glance at the dynamic programming principle: roughly speaking in the Bellman principle on east step we choose the worst scenario, so the less steps the less risk. The example on previous slide demonstrates this.

Is it good?

1. I'm not sure. Anyway it's worth to describe this in mathematical terms in order to know which properties can live together and which can't.

2. I can suggest an economical reasoning explaining this phenomenon. *The problem with the information about our final performance, arriving in intermediate dates, is that it might not just inform us about bad expectations,* **it might inform the others about bad expectations for us**. *Thus it might hit us in an indirect way.*

Let's go back to the question if there is a natural law invariance for dynamic risk measures.

We say that a process $X \in \mathcal{R}^{\infty}_{1,T}$ is "simple" if $X = X_{t_0}\mathbf{1}_{t_0} =$ with some t_0 (i.e. there is only 1 non-zero cash-flow) and X_{t_0} is **independent of** $\mathcal{F}_{t_0-1}$.

It's still appears (at least to me) that the following properties for "simple" process are natural:

LI1 If X and Y are "simple" (with the same t_0) and $\mathrm{Law}(X_{t_0}) = \mathrm{Law}(Y_{t_0})$, then

$$u_t(X) = u_t(Y) \qquad \forall t.$$

LI2 If X is simple, then $u_{t_0-1}(X)$ is constant (non random). (*It's natural since due to the independence $Law(X_{t_0}|\mathcal{F}_{t_0-1}) = Law(X_{t_0})$ so no new information about the law of future cash-flows appear from $t = 0$ to $t = t_0$.*)

It is easy to check that DTVaR^{λ} *with non-random* λ satisfies both LI1 and LI2. DTVaR^{λ} with random λ does not satisfy LI2.

For an abstract concave dynamic time-consistent monetary utility functional u (with Fatou property etc) satisfying LI1 and LI2, it is straightforward to have the following extension of the Kusuoka's formula:

$$u_t(X) = \widetilde{V}^t\Big[\mathsf{Law}\Big(X_{t+1} + u_{t+1}(X)\Big|\mathcal{F}_t\Big)\Big], \qquad (*)$$

where $\widetilde{V}^t$, $t = 0, \dots, T-1$ are functionals, defined on probability laws, *generated by some classical 1-step convex monetary law-invariant utilities (so we can apply the classical Kusuoka's formula to characterize it).*

Vice versa, if we have some $\widetilde{V}^t$, $t = 0, \dots, T-1$ as above, then (*) defines a concave dynamic time-consistent monetary utility functional u (with Fatou property etc) satisfying LI1 and LI2.

If forget about a couple of technicalities it is very simple result. To be candid it is almost the case that we assumed (*) and proved it's equivalence to LI1 and LI2 — two very simple properties, *related to law invariance*.

But what is a true law-invariance?
If (*) holds and we fix $X \in \mathcal{R}^{\infty}_{1,T}$, is it possible to describe the class

$$\{Y \sim X\} = \{Y \in \mathcal{R}^{\infty}_{1,T} : u_t(Y) = u_t(X) \ \forall t \ \forall u \text{ satisfying LI1 and LI2}\}? \qquad (**)$$

I can't.

So far *the "dynamic law-invariance" in our framework is a property of dynamic concave monetary utilities, not of cash-flow processes.* This "dynamic law-invariance" (i.e. axioms LI1 and LI2) looks quite natural because LI1 and LI2 look natural. Of course this "dynamic law-invariance" implies a kind of law-invariance for processes via (**).

Unfortunately, we cannot find a good description of law-invariance for processes.

At first naïve approach one can think that it is natural to have the following version of invariance:
If the cash flow processes $X, Y \in \mathcal{R}^{\infty}_{1,T}$ satisfy

$$\mathsf{Law}(X_t, \ldots, X_T | \mathcal{F}_{t-1}) = \mathsf{Law}(Y_t, \ldots, Y_T | \mathcal{F}_{t-1}), \qquad \forall t, \qquad (***)$$

then we should estimate the risk of them identically: $u_t(X) = u_t(Y)$.

It is not the case because of the dynamic programming principle

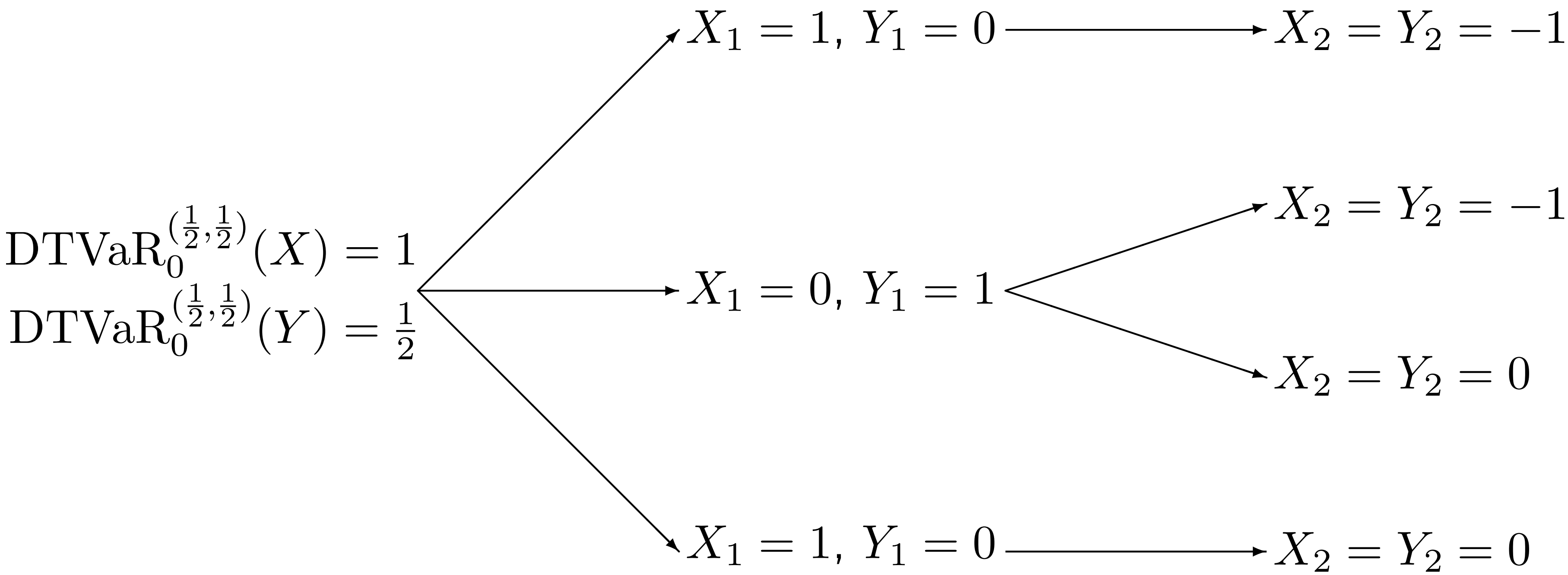
$$u_t(X) = u_t\big((X_{t+1} + u_{t+1}(X))\mathbf{1}_{\{t+1\}}\big).$$

Roughly: even if we have $\mathsf{Law}(X_{t+1}|\mathcal{F}_t) = \mathsf{Law}(Y_{t+1}|\mathcal{F}_t)$ and $\mathsf{Law}(u_{t+1}(X)|\mathcal{F}_t) = \mathsf{Law}(u_{t+1}(Y)|\mathcal{F}_t)$ we could have

$$\mathsf{Law}(X_{t+1} + u_{t+1}(X)|\mathcal{F}_t) \neq \mathsf{Law}(Y_{t+1} + u_{t+1}(Y)|\mathcal{F}_t).$$

We constructed a simple example to show that our best candidates for "dynamic law-invariant" risk measures: DTVaR^{λ} with constant λ are not invariant w.r.t. to the invariance implied by (***).

Example: Let $T = 2$, Ω consists of 4 equimeasurable atoms (no problem to reformulate for atomless) and the stochastic structure is given by the tree is



It it easy to check that

$$\mathsf{Law}(X_1, X_2) = \mathsf{Law}(Y_1, Y_2) \qquad \text{and} \qquad \mathsf{Law}(X_2|\mathcal{F}_1) = \mathsf{Law}(Y_2|\mathcal{F}_1).$$

But $1 = \mathrm{DTVaR}_0^{(\frac{1}{2},\frac{1}{2})}(Y) \neq \mathrm{DTVaR}_0^{(\frac{1}{2},\frac{1}{2})}(Y) = \frac{1}{2}$.

Conclusions:

1. For dynamic time-consistent convex risk measures we introduced 2 natural axioms relevant to law-invariance: LI1 and LI2.

2. We proved that LI1 and LI2 hold iff in the dynamic programming principle the operators transferring $t+1 \to t$ depend only on the conditional law: Law$(X_{t+1} + u_{t+1}(X)|\mathcal{F}_t)$.

3. We also demonstrated that in the dynamic setting naïve law invariance should not be considered as a natural properties.

4. We noticed that some implications of strong time-consistency should be discussed further from economical point of view.

Thanks!

Danke!