SPDES driven by Poisson Random Measures

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Motivations of SPDEs

- Finance Mathematics: The forward interest rate of a zero bound in the Heath Jarrow Morton model is is described by a SPDE driven by Wiener or Lévy noise;
- Physics: in thin-film models, SPDEs leads to a better description of data's gained by experiments [Grüne, Mecke, Rauscher (2006)];
- Physics: Falkovich, Kolokolov, Lebedev, Mezentsev, and Turitsyn (2004) uses stochastic nonlinear Schrödinger equation to describe certain parameters in optical soliton transmission;
- Population dynamics ....
- Biology ....

Outline of the talk

- An Example from Finance
- Lévy processes - Poisson Random Measure
- SPDEs driven by Poisson Random Measure
  - Existence and Uniqueness Results
- Further Works and Open Questions

Heath Jarrow Morton Model (1992):

A zero coupon bond with maturity date T is a contract which guarantees the holder 1 Dollar to be paid at time T.

- $p(t,x)$: Price at time t of a zero coupon bond maturing at time $t+x$;
- $r(t,x)$: Forward rate, contracted at t, maturing at time $t+x$;
- $R(t)$: Short interest rate;

$$\begin{cases} r(t,x) &= -\frac{\partial \log p(t,x)}{\partial x} \\ p(t,x) &= \exp\left(-\int_0^x r(t,s)\,ds\right); \\ R(t) &= r(t,0). \end{cases}$$

Heath Jarrow Morton Model (1992):

The HJM-Model describes the dynamic of the forward interest rate under the assumption that the bond market is free of arbitrage. In particular, the forward rate function solves the following SPDE

$$\begin{cases} dr(t,x) &= \left[\frac{\partial}{\partial x} r(t,x) + f(t,x)\right] dt + \sum_{k=1}^{\infty} \sigma^k(t,x) dw_k(t), \; x \geq 0; \\ r(t,0) &= R(t), \quad x \geq 0; \end{cases}$$

where w_k, $k \in \mathbb{N}$, are real valued independent Wiener processes and f satisfies the well–known HJM drift condition

$$f(t,x) = \sum_{k=1}^{\infty} \sigma^k(t,x) \int_0^x \sigma^k(t,y)\, dy.$$

Talk of Eberlein on monday morning;
Björk et. all (1997); Filipovic (2001); Ben Goldys and Musiela (2001);
...

HJM Model with Lévy noise:

The SPDE of the corresponding model with Lévy noise is given by

$$\begin{cases} dr(t,x) &= \left[\frac{\partial}{\partial x} r(t,x) + f(t,x)\right] \, dt + b(t) dL(t), \quad x \geq 0; \\ r(t,0) &= R(t), \quad x \geq 0; \end{cases}$$

where L is an infinite dimensional Lévy processes taking values in a certain Hilbert space and f satisfies the HJM drift condition.

References for the HJM condition: Björk, Di Masi, Kabanov and Runggaldier (1997); Björk, Kabanov and Runggaldier (1997); Eberlein, Jacod and Raible (2005); Peszat and Zabczyk (2007).

Further References: Albeverio, Lytvynov and Mahnig (2004); Eberlein and Raible (1999); Jakubowski and Zabczyk (2007, 2004); Rusinek (2006); Marinelli (2006); Tappe (2007) (Talk on friday).

A typical Example

We are interested in SPDEs of the following type:

$$\begin{cases} du(t,\xi) &= \nabla u(t-,\xi)\, dt + g(u(t-,\xi))dL(t) \\ & \quad + f(u(t-,\xi))\, dt, \qquad \xi \geq 0,\ t > 0; \\ u(0,\xi) &= u_0(\xi) \quad \xi \geq 0; \end{cases}$$

where $u_0 \in L^p(0,1)$, $p \geq 1$, g a certain mapping and $L(t)$ is a Lévy process specified later.

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The Abstract Cauchy Problem

Linear evolution equations, as parabolic, hyperbolic or delay equations, can often be formulated as an evolution equation in a Banach space E:

Given:

- E Banach space,
- the pair $(A, \text{dom}(A))$, where $A : E \to E$ a linear, in general unbounded, operator defined on a dense linear subspace dom(A) of E;
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Problem: The solution to the following initial valued problem:

$$\begin{cases} u'(t) &= A\,u(t), \quad t \geq 0, \\ u(0) &= u_0 \in E. \end{cases}$$

The Wave Equation:

Example 1

$$(\star) \quad \begin{cases} \frac{d}{dt}u(t,\xi) &= \nabla u(t,\xi), \quad t>0,\ \xi\ge 0; \\ u(0,\xi) &= u_0(\xi), \quad \xi \ge 0; \end{cases}$$

The solution of the Cauchy problem $(\star)$ is given by the shift semigroup. In particular, let $(S(t))_{t\ge 0}$ be defined by

$$S(t)u(x) := u(t+x), \quad u \in \mathcal{C},$$

then $u(t) := S(t)u_0$ is a solution to $(\star)$.

The Laplace Operator

Example 2 *In one of the first slides we had the following example: Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.*

$$\begin{cases} \frac{du(t,\xi)}{dt} & = & \Delta\, u(t,\xi), \quad t>0,\ \xi\in\mathcal{O}; \\ u(0,\xi) & = & u_0(\xi), \quad \xi\in\mathcal{O}; \\ u(t,\xi) & = & 0, \quad t\geq 0;\ \xi\in\partial\mathcal{O} \end{cases}$$

Formulated in semigroup theory, $(\star)$ gives the following Cauchy problem:

$$\begin{aligned} E &:= L^2(\mathcal{O}) \quad \text{or } L^p(\mathcal{O}),\ 1<p<\infty, \\ A &= \Delta, \quad u(0)=u_0; \\ \text{dom}(A) &:= \left\{u\in L^2(\mathcal{O}),\ Au\in L^2(\mathcal{O}),\ u\big|_{\partial\mathcal{O}}=0\right\}. \end{aligned}$$

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Problem: The solution to the following initial valued problem:

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The Abstract Cauchy problem:

The Cauchy Problem is well posed if:

- for arbitrary $u_0 \in$ dom(A) there exists exactly one strong differentiable function $u(t, u_0)$, $t \geq 0$ satisfying $(\star)$ for all $t \geq 0$.
- if $\{x^n\} \in$ dom(A) and $\lim_{n\to\infty} x_n = 0$, then for all $t \geq 0$ we have

$$\lim_{n\to\infty} u(t, x_n) = 0.$$

The Abstract Cauchy problem:

Assume a solution exists and let us define the linear operator $S(t)$: dom$(A) \to E$ by the formula

$$S(t)x = u(t, u_0), \quad \forall u_0 \in \text{dom}(A),\ \forall t \geq 0.$$

The family of operators $S(\cdot)$ can be extended to an operator on E. Moreover, we have

$$S(0) = I, \quad S(t+s) = S(t)S(s); \quad \forall t, s \geq 0.$$

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Definition 1 *A semigroup $S(t)$, $0 \leq t < \infty$ of bounded linear operators on E is a strongly continuous semigroup (C_0- semigroup) if*

$$\lim_{t \to 0} S(t)x = x, \quad \textit{for every } x \in E.$$

The Infinitesimal Generator of a Semigroup

Definition 1 *The infinitesimal generator of a semigroup* $S(\cdot)$ *is a linear operator defined by*

$$\begin{cases} \textit{dom}(A) & := & \left\{ x \in E : \exists \lim_{h \to 0^+} \frac{S(h)x - x}{h} \right\} \\ Ax & := & \lim_{h \to 0^+} \frac{S(h)x - x}{h}, \quad \forall x \in \textit{dom}(A). \end{cases}$$

Variation of Constants Formula

The Abstract Problem: Given $f \in L^1([0,T];E)$. We ask for a solution to

$$(\bullet) \quad \begin{cases} u'(t) &= Au(t) + f(t); \\ u(0) &= x \in E. \end{cases}$$

The solution is given by the variation of constant formula

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds, \quad t \in (0,T].$$

and is called the *mild solution* to $(\bullet)$.

A typical Example

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where $u_0 \in L^p(0,1)$, $p \ge 1$, g a certain mapping and $L(t)$ is a Lévy process specified later.

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We are interested in SPDEs of the following type:

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The Lévy Process L

Let E be a Banach space. Assume that $L = \{L(t), 0 \le t < \infty\}$ is a E–valued Lévy process over $(\Omega; \mathcal{F}; \mathbb{P})$. Then L has the following properties:

- $L(0) = 0$;
- L has independent and stationary increments;
- for ϕ bounded, the function $t \mapsto \mathbb{E}\phi(L(t))$ is continuous on $\mathbb{R}^+$;
- L has a.s. cádlág paths;
- the law of $L(1)$ is infinitely divisible;

The Lévy Process L

E denotes a separable Banach space and E' the dual on E. The Fourier Transform of L is given by the **Lévy - Hinchin - Formula**:

$$\mathbb{E}\, e^{i\langle L(1),a\rangle} =$$

$$(\star) \qquad \exp\left\{ i\langle y,a\rangle\lambda + \int_E \left(e^{i\lambda\langle y,a\rangle} - 1 - i\lambda y 1_{\{|y|\le 1\}} \right) \nu(dy) \right\},$$

where $a \in E'$, $y \in E$ and $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is a certain measure.

We call these symmetric measures $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ for which $(\star)$ is well defined symmetric Lévy measures. If ν is a σ–finite measure and its symmetrisation is a symmetric Lévy measure, we call it Lévy measure (see Linde (1986)).

Poisson Random Measure

For any $A \in \mathcal{B}(E)$, the so-called counting measure can be defined by

$$N(t, A) = \sharp \{s \in (0, t] : \Delta L(s) = L(s) - L(s-) \in A\}.$$

One can show, that

- $N(t, A)$ is a random variable over $(\Omega; \mathcal{F}; \mathbb{P})$;
- $N(t, A) \sim$ Poisson $(t\nu(A))$ and $N(t, \emptyset) = 0$;
- For any pairwise disjoint sets $A_1, \ldots, A_n$, the random variables $N(t, A_1), \ldots, N(t, A_n)$ are pairwise independent;

Poisson Random Measure

Definition 2 *Let $(S, \mathcal{S})$ be a measurable space and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. A* random measure *on $(S, \mathcal{S})$ is a family*

$$\eta = \{\eta(\omega, \cdot), \omega \in \Omega\}$$

of non-negative measures $\eta(\omega, \cdot) : \mathcal{S} \to \mathbb{N}_0$, such that

- $\eta(\cdot, \emptyset) = 0$ *a.s.*
- η *is a.s. σ–additive.*
- η *is independently scattered, i.e. for any finite family of pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$, the random variables*

  $$\eta(\cdot, A_1), \ldots, \eta(\cdot, A_n)$$

  are pairwise independent.

Poisson Random Measure

A random measure η on $(S,\mathcal{S})$ is called Poisson random measure iff for each $A \in \mathcal{S}$ such that $\mathbb{E}\,\eta(\cdot,A)$ is finite, $\eta(\cdot,A)$ is a Poisson random variable with parameter $\mathbb{E}\,\eta(\cdot,A)$.

Remark 1 *The mapping*

$$\mathcal{S} \ni A \mapsto \nu(A) := \mathbb{E}\,\eta(\cdot,A) \in \mathbb{R}$$

is a measure on $(S,\mathcal{S})$*.*

Poisson Random Measure

Let $(Z, \mathcal{Z})$ be a measurable space. If $S = Z \times \mathbb{R}^+$, $\mathcal{S} = \mathcal{Z} \hat{\times} \mathcal{B}(\mathbb{R}^+)$, then a Poisson random measure on $(S, \mathcal{S})$ is called Poisson point process.

Remark 2 *Let ν be a Lévy measure on a Banach space E and*

- $S = E \times \mathbb{R}^+$
- $\mathcal{S} = \mathcal{B}(E) \hat{\times} \mathcal{B}(\mathbb{R}^+)$
- $\nu' = \nu \times \lambda$ *(λ is the Lebegues measure).*

Then there exists a time homogeneous Poisson random measure

$$\eta : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

such that $\mathbb{E}\, \eta(\,\boldsymbol{.}\,, A, I) = \nu(A)\lambda(I), \quad A \in \mathcal{B}(E), I \in \mathcal{B}(\mathbb{R}^+)$*,*
ν is called the intensity of η.

Poisson Random Measure

$$(\star) \quad t \mapsto \int_0^t \int_E z \;\; \eta(dz, ds)$$

Poisson Random Measure

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Remark 3 *The integral in* $(\star)$ *is well defi ned if the intensity of* η *is a symmetric Lévy measure (and* E *a certain Banach space).*

Poisson Random Measure

Definition 2 *Let*

$$\eta : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

be a Poisson random measure over $(\Omega; \mathcal{F}; \mathbb{P})$ *and* $\{\mathcal{F}_t, 0 \le t < \infty\}$ *the filtration induced by* η*. Then the predictable measure*

$$\gamma : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

is called compensator of η*, if for any* $A \in \mathcal{B}(E)$ *the process*

$$\eta(A, (0,t]) - \gamma(A, [0,t])$$

is a local martingale over $(\Omega; \mathcal{F}; \mathbb{P})$*.*

Remark 3 *The compensator is unique up to a* $\mathbb{P}$*-zero set and in case of a time homogeneous Poisson random measure given by*

$$\gamma(A, [0,t]) = t\,\nu(A), \quad A \in \mathcal{B}(E).$$

Poisson Random Measure

$$(\star) \quad t \longmapsto \int_0^t \int_E z \underbrace{(\eta - \gamma)}_{:=\tilde{\eta}} (dz, ds)$$

Poisson Random Measure

$$(\star) \quad t \mapsto \int_0^t \int_E z \underbrace{(\eta - \gamma)}_{:=\tilde{\eta}} (dz, ds)$$

Remark 4 *The integral in $(\star)$ is well defined if the intensity of η is a Lévy measure (and E a certain Banach space).*

Poisson Random Measure

Let L be a E-valued Lévy process and let again $N(t,\cdot)$ be the counting measure given by

$$\mathcal{B}(E) \ni A \mapsto N(t,A) := \sharp \left\{ s \in (0,t] : \Delta L(s) = L(s) - L(s-) \in A \right\}.$$

For any interval $I = (s,t]$, let $\eta(\cdot, I) : \mathcal{B}(E) \to \mathbb{N}^0$ be defined by

$$\mathcal{B}(E) \ni A \mapsto \eta(A, I) := N(t,A) - N(s,A).$$

Then the extension of η to $\mathcal{B}(E) \hat{\times} \mathcal{B}(\mathbb{R}^+)$ gives a Poisson random measure.

A typical Example

We are interested in SPDEs of the following type:

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where $u_0 \in L^p(0,1)$, $p \geq 1$, g a certain mapping and $L(t)$ is a Lévy process taking values in a certain Banach space.

Banach spaces of M type p

Definition 3 *(see e.g. Pisier (1986)) Let $1 \le p < \infty$. A Banach space E is of M type p (or uniformly p integrable) , iff there exists a constant $C = C(E;p)$, such that for each discrete E-valued martingale $M = (M_1, M_2, \ldots)$ one has*

$$\sup_{n\ge 1} \mathbb{E}|M_n|_E^p \le C \sum_{n\ge 1} \mathbb{E}|M_n - M_{n-1}|_E^p.$$

Remark 5 *A Banach space is uniformly p convex if there exists a equivalent norm $\|\cdot\|$ in E, such that*

$$\frac{1}{2}\left(|x+y|_E^p + |x-y|_E^p\right) \le |x|^p + \|y\|_E^p.$$

Pisier has shown, that if a Banach space E is uniformly p convex then E is of M–type p.

Banach spaces of M type p

Example 3 *(see e.g. Linde (1986), Chapter 2) If* $(M, \mathcal{M}, \mathbb{P})$ *is a probability space and* $p > 1$*, then the space* $L^p(M, \mathcal{M}, \mathbb{P})$ *is of* M*-type* $p \wedge 2$*.*

Example 4 *Let* $(S, \mathcal{S})$ *be a measurable space. Then* $L^\infty(S)$*,* $L^1(S)$ *are often not M type. The space* $\mathcal{C}([0,1];\mathbb{R})$ *is* **not** *of M type p.*

Burkholder inequality

Proposition 1 *Let E be a Banach space of M-type p, $1 < p \leq 2$. Then there exists a constant $C = C(E; p) < \infty$, such that we have for any discrete E-valued martingale $M = (M_1, M_2, \ldots)$ and for all $1 \leq r < \infty$*

$$\mathbb{E} \sup_{n \geq 1} |M_n|_E^r \leq C \mathbb{E} \left[\sum_{n \geq 1} |M_{n-1} - M_n|_E^p \right]^{\frac{r}{p}} .$$

The Itô Stochastic Integral

In M-type p Banach spaces on can define the stochastic integral with respect to Lévy processes by the extension procedure:

Let h be a càglàd step function given by

$$h(t) = \sum_{i=1}^{n} H_i 1_{(t_i,t_{i+1}]}(t), \quad 0 \le t \le T,$$

where $0 = t_0 \le \cdots t_n = T$ and $H_i : \Omega \to L(Z, E)$ is $\mathcal{F}_{t_i}$–measurable, $i = 1, \ldots, n$.

Definition 4 *The* *stochastic integral* *of h with respect to η is defined by*

$$I(h) := \sum_{i=1}^{n} \int_Z H_i(z)\eta(dz; (t_i, t_{i+1}]). \quad (\spadesuit)$$

Definition of the Integral

Let $\mathcal{M}^p([0,T];E)$ be the space of all predictable functions $h:[0,T]\times\Omega\to L(Z,E)$ such that

$$\int_0^T\int_Z \mathbb{E}|h(s,z)|_E^p\,\nu(dz)\,ds<\infty.$$

Theorem 1 *There exists a linear bounded operator*

$$I:\mathcal{M}^p([0,T];E)\to L^p(\Omega,\mathcal{F}_T,\mathbb{P};E),$$

which is a unique bounded extension of the operator defined in (♠). If $h\in\mathcal{M}^p([0,T];E)$ and $t>0$ then we put

$$\int_{0+}^t\int_Z h(s,z)\,\eta(dz;ds):=I(1_{(0,t]}h)$$

and we call the LHS the Itô integral of the process h up to time t.

Properties of the Stochastic Integral

- If $h \in \mathcal{M}^p([0,T];E)$, then the process

$$X(t) = \int_{0+}^{t} \int_Z h(s,z)\,\eta(dz;ds), \quad t \geq 0$$

  is an E–valued martingale having a càdlàg modification.

- There exists a constant $C = C(p,E) < \infty$, such that for any $h \in \mathcal{M}^p([0,T];E)$ and for any $0 < r \leq p (\leq 2)$

$$\mathbb{E} \sup_{0<t\leq T} \left| \int_{0+}^{t} \int_Z h(s,z)\,\eta(dz;ds) \right|^r \leq$$

$$C \left(\int_{0+}^{T} \int_Z \mathbb{E}\,|h(s,z)|_E^p\,\nu(dz)\,ds \right)^{\frac{r}{p}}.$$

Some References:

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A typical Example

We are interested in SPDEs of the following type:

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where $u_0 \in L^p(0,1)$, $p \geq 1$, g a certain mapping and $L(t)$ is a Lévy process specified later.

SPDES - the Abstract Form

Let E be Banach space of M–type p and let A be the infinitesimal generator of an analytic semigroup in E. Our interest lies in the following SPDE written in the Itô-form

$$(1) \quad \begin{cases} du(t) &= Au(t-)\,dt + f(u(t))\,dt + \int_Z g(u(t-);z)\tilde{\eta}(dz;dt), \\ u(0) &= u_0 \in E. \end{cases}$$

A mild solution of equation (1) is an adapted E-valued càdlàg process $u = \{u(t) : t \in [0,T]\}$ such that for $t \geq 0$

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\, f(u(s))\,dt +$$

$$\int_{0+}^t \int_Z S(t-s)g(u(s-);z)\,\tilde{\eta}(ds,dz), \text{ a.s. .}$$

SPDEs - Existence and Uniqueness

Theorem 2 *(EH, 2005 EJP) Let E be Banach space of M–type p, $B \hookrightarrow E$ compactly. Assume that*

- $\mathbb{E}|u_0|_B^p < \infty$;
- *there exists some $\delta_f < 1$ such that $(-A)^{-\delta_f} f : E \to E$ is Lipschitz continuous;*
- *there exists some $\delta_g < \frac{1}{p}$ such that $(-A)^{-\delta_g} g : E \to L(Z, E)$ satisfies*

$$\int_Z \left| (-A)^{-\delta_g} \left(g(x,z) - g(y,z) \right) \right|^p \nu(dz) \le C \, |x-y|^p, \quad x, y \in E.$$

Then, there exists a unique mild solution to Problem (1) such that for any $T > 0$

$$\int_0^T \mathbb{E}|u(s)|^p \, ds < \infty,$$

and $(-A)^{-\delta_0} u \in L^0(\Omega; \mathbb{D}([0,T]; E))$ for some $\delta_0 > \delta_g, \delta_f$.

Outline of the Proof of Theorem 1:

One starts with the following space

$$\mathcal{V}_p := \left\{ (-A)^{-\delta_0} u : \Omega \to \mathbb{D}([0,T];E),\ \int_0^T \mathbb{E}|u(s)|^p\, ds < \infty \right\}$$

with norm

$$\|u\|_{\mathcal{V}_p} := \left(\int_0^T \mathbb{E}|u(s)|^p\, ds \right)^{\frac{1}{p}}.$$

Again, let $\overline{\mathcal{V}_p}$ be the completion of $\mathcal{V}_p$.

Remark 6 *If $\delta_0 > 0$ then the set $\mathcal{V}_p$ is a proper subset of $\overline{\mathcal{V}_p}$.*

Outline of the Proof of Theorem 2:

First, we define for a fixed u_0 the operator

$$
\begin{aligned}
(\mathcal{K}_{u_0} u)(t) = S(t)u_0 &+ \int_{0+}^{t} S(t-s) f(u(s-)) ds \\
&+ \int_{0+}^{t} \int_Z S(t-s) g(u(s-); z) \tilde{\eta}(dz; ds), \quad t \in [0, T]
\end{aligned}
$$

and then we show the following Lemma:

Lemma 1 *For any* $u_0 \in V_\gamma$

- *the operator* $\mathcal{K}_{u_0}$ *maps* $\mathcal{V}_p$ *into* $\mathcal{V}_p$ *and*
- *the operator* $\mathcal{K}_{u_0}$ *is for* T *small enough a contraction.*

Outline of the Proof of Theorem 2:

Suppose, $T > 0$ is so small, such that $\mathcal{K}_{u_0} : \mathcal{V}_p \to \mathcal{V}_p$ is a contraction. Then again follows, that for each $u_0 \in V_\gamma$ there exists a unique $u^* \in \overline{\mathcal{V}_p}$,

such that

$$\mathcal{K}_{u_0} u^* = u^*$$ [a]

and

$$\mathcal{K}_{u_0}^{(n)} v \longrightarrow u^*$$

for all $v \in \mathcal{V}_p$.

Finally we have to show, that $u^\star \in \mathcal{V}_p$. But since $\mathcal{V}_p$ is a proper subset of $\overline{\mathcal{V}_p}$, it is not trivial to show $(-A)^{-\delta_0} u^\star \in L^0(\Omega; \mathbb{D}([0,T]; E))$.

[a] Note, that $\mathcal{K}_{u_0}$ is defi ned on $\overline{\mathcal{V}_p}$ by extension.

SPDEs of Reaction Diffusion Type

We are interested in SPDEs of the following type:

$$(\Diamond)\quad \begin{cases} du(t) &= \big(\Delta u(t-) - u^3(t-) + u(t-)\big)\,dt + dL(t), \quad t \geq 0, \\ u(0,\xi) &= u_0(\xi) \quad 0 \leq \xi \leq 1, \\ u(t,0) &= u(t,1) = 0, \quad t \geq 0, \end{cases}$$

where $u_0 \in L^p(0,1)$, $p \geq 1$, and $L(t)$ is a Lévy process.

Or an SPDE given by

$$(\clubsuit)\quad \begin{cases} du(t) &= Au(t-)\,dt + F(t-, u(t-))\,dt \\ & \qquad + \int_Z G(t-, u(t-); z)\eta(dz; dt), \\ u(0) &= u_0 \in E, \end{cases}$$

where F and G are not global Lipschitz, but continuous and bounded, E is a Banach space.

Solution of Martingale Type

Definition 5 *A martingale solution to equation (♣) is a system*

$$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \{\tilde{\eta}(t,z)\}_{t\geq 0, z\in Z}, \{u(t)\}_{t\geq 0})$$

such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration on it, $\{\eta(t,z)\}_{t\geq 0, z\in Z}$ is a time homogeneous Poisson Random measure on $\mathcal{B}(Z)\times\mathcal{B}(\mathbb{R}^+)$ over $(\Omega, \mathcal{F}, \mathbb{P})$ (with respect to the filtration $\mathcal{F}_t$) with intensity ν and $u(t)$ is a B–valued adapted process such that for any $t\in[0,T]$

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(s,u(s))\,ds$$
$$+\int_0^t\int_Z e^{-(t-s)A}G(s,u(s-);z)\,d\tilde{\eta}(dz,ds),\ a.s..$$

Work in Progress with Brzezniak.

References - Similar Results:

Books:

- Forthcoming book of Zabczyk and Peszat,
- Metivier: SPDEs in infi nite-dimensional spaces (1988)

Articles:

- Albeverio, Wu and Zhang (1998); Applebaum and Wu (2000); St. Lubert Bié (1998); Kallianpur and Xiong (1987); Knoche (2006); Fournier (2001) [Support theorem]; Fournier (2000) [Malliavin Calculus] León and Sarrá, (2002); Röckner, Zhang (2007); Röckner and Lescot (2004);

Further References

- **Properties of the Ornstein–Uhlenbeck process:**

  Chojnowska-Michalik (1987) (she looked also for the invariant measure); Applebaum (2006,2007) Röckner and Zhang (2007); Rusinek (2006); Seidler and H. (2001,2007);

- **Numeric of SPDEs:**

  Li, Pang and Wang (2007); Marchis and E.H.(2006); E.H.(2007) (Approximation by Finite Elements) ; H. (2007) (Wong Zakai Approximation); Kouritzin, Long and Sun (2003);

- **SPDEs with nonlinear but bounded perturbation:**

  Mytnik (2002); Mueller (1999); Mueller, Mytnik and Stan (2006); Hausenblas (2007);

Some References

- **SPDEs with unbounded nonlinearity:**

  Gugg and Duan (2004); Truman and Wu (2006) [Fractals Burger equation]; Dong and Xu (2007) [Burger equation]; Bo and Wang (2006) [Cahn Hillard Equation]

- **SPDEs tackled by Wick Products:**

  Løkka, Øksendal, Proske (2004); Dermoune (1997);

- **None above but in this section:**

  Kallianpur and Xiong (1994); Cranston, Mountford and Shiga (2002) [Andersen Model with Lévy noise]; Walsh (1981); . . .

Further Works

- Numerical Approximation
- Wong Zakai Approximation
- Existence of Invariant Measure
- Uniqueness of the Invariant Measure
- Strong Feller Property of the Ornstein Uhlenbeck process
- Strong Feller Property of an Arbitrary Solution
- Different Type of Equations
- ...
- ...

The End