A structural multi issuer credit risk model based on square root processes

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Default correlation

We want to compute the default correlation $\rho(t)$, i.e. the correlation of the default indicators $\mathbb{1}_{\{T_i \leq t\}}$ and $\mathbb{1}_{\{T_j \leq t\}}$ for $i \neq j$ where T_i and T_j are default times.
We get

$$\begin{aligned}
\rho(t) &= \frac{\mathbb{E}\left(\mathbb{1}_{\{T_i \leq t\}}\mathbb{1}_{\{T_j \leq t\}}\right) - \mathbb{E}\left(\mathbb{1}_{\{T_i \leq t\}}\right)\mathbb{E}\left(\mathbb{1}_{\{T_j \leq t\}}\right)}{\sqrt{Var\left(\mathbb{1}_{\{T_i \leq t\}}\right)Var\left(\mathbb{1}_{\{T_j \leq t\}}\right)}} \\
&= \frac{\mathbb{P}(T_i \leq t, T_j \leq t) - \mathbb{P}(T_i \leq t)\mathbb{P}(T_j \leq t)}{\sqrt{\mathbb{P}(T_i \leq t)(1 - \mathbb{P}(T_i \leq t))\mathbb{P}(T_j \leq t)(1 - \mathbb{P}(T_j \leq t))}}
\end{aligned}$$

Default time

In a first passage time model the default time T_i is defined as the first time an (possibly artificial) ability-to-pay process $(X_t^i)_{t \geq 0}$ hits a certain barrier K_i, i.e.

$$T_i = \inf\{t \geq 0 : X_t^i \leq K_i\}$$

Later on this barrier K_i will be equal to zero for all obligors i.

Ability to pay process

The ability to pay process of company i is a stochastic process $X^i = (X^i_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
$\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of the processes $X^i, i \in \mathcal{I}$, i.e. $\mathcal{F}_t = \sigma(\{X^i_s\}_{i \in \mathcal{I}}, 0 \leq s \leq t)$. The process X^i_t is given by

$$X^i_t = Y_t + Y^i_t$$

Ability to pay process

The processes Y_t and $\{Y_t^i\}_{i \in \mathcal{I}}$ are independent. The process Y_t^i represents the idiosyncratic default risk of obligor i and Y_t is a common process that affects all obligors in equal measure.
A similar concept, though not as ability to pay processes, has already been applied in reduced form models, see (Duffie & Singleton, 2003)

Reduced form model

In a reduced form model the default probability is given as

$$\mathbb{P}(\tau \leq t) = 1 - \mathbb{E}\left(\exp\left(-\int_0^t \lambda(u)du\right)\right)$$

The intensity process $\lambda(t)$ is modelled as a stochastic process with state space $\mathbb{R}^+$ satisfying

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t) + \Delta J(t)$$

Reduced form model

A process $\lambda(t)$ satisfying

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t) + \Delta J(t)$$

is called a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$, where μ is the mean of the exponentially distributed jump sizes and l is the mean jump arrival rate of the pure jump process J.

Reduced form model

In order to introduce dependencies between different obligors, Duffie proposed to describe the intensity process λ_i of obligor i as the sum of two basic affine processes, i.e.

$$\lambda_i = X_c + X_i$$

X_c has the parameters $(\kappa, \theta_c, \sigma, \mu, l_c)$, X_i has the parameters $(\kappa, \theta_i, \sigma, \mu, l_i)$. Then λ_i is itself a basic affine process with parameters $(\kappa, \theta, \sigma, \mu, l)$ where $\theta = \theta_c + \theta_i$ and $l = l_c + l_i$.

Squared Bessel process

The special case $l = 0$, i.e. with no jumps at all leads to the Cox-Ingersoll-Ross process

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t)$$

We concentrate on a certain process from the Cox-Ingersoll-Ross family, namely the Squared Bessel process.

Squared Bessel process

The Squared Bessel process is defined as follows

Definition

For every $\delta \geq 0$ and $x_0 \geq 0$ the unique strong solution to the equation

$$X_t = x_0 + \delta t + 2\int_0^t \sqrt{X_s} dB_s$$

where B_t is a standard Brownian motion, is called δ-dimensional squared Bessel process started at x_0 and is denoted by $BESQ^{\delta}_{x_0}$.

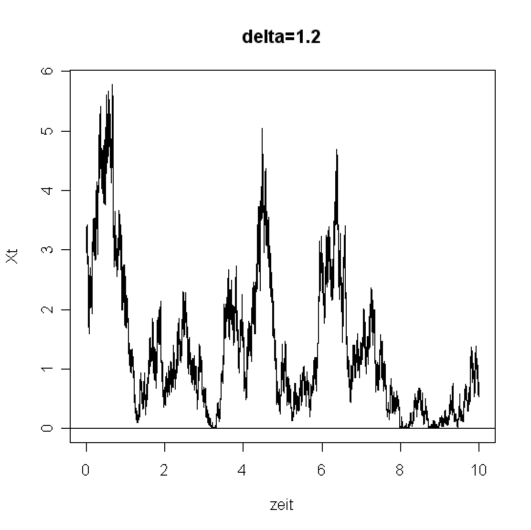
Squared Bessel process

Behavior of the trajectories of squared Bessel processes:

- $\delta = 0$: The point $x = 0$ is absorbing, (after it reaches 0, the process will stay there forever).
- $0 < \delta < 2$: The point $x = 0$ is reflecting, (the process immediately moves away from 0).
- $\delta \geq 2$: The point $x = 0$ is unattainable.

Squared Bessel process

Figure: $BESQ_3^{1.2}$



First hitting time

As already mentioned, the default barrier is set to zero. Therefore we need to compute the first hitting time of zero of a Squared Bessel process of dimension $\delta \in (0, 2)$, since for $\delta = 0$, the only free parameter is the starting value and for $\delta \geq 2$ the process never reaches zero.

The density of the first hitting time of a $BESQ^{\delta}_{x_0}$ in zero is given as

$$\mathbb{P}(T_0 \in dt) = \frac{1}{t\Gamma(1 - \frac{\delta}{2})} \left(\frac{x_0}{2t}\right)^{1-\frac{\delta}{2}} e^{-\frac{x_0}{2t}} dt$$

First hitting time

Thus, the first hitting time T_0 of a $BESQ^{\delta}_{x_0}$ is inverse Gamma distributed, i.e.

$$T_0 \sim Inv - Gamma\left(1 - \frac{\delta}{2}, \frac{x}{2}\right)$$

This means

$$\frac{1}{T_0} \sim Gamma\left(1 - \frac{\delta}{2}, \frac{x}{2}\right)$$

Squared Bessel process

Due to the additivity property of squared Bessel processes, the process $Y_t + Y_t^i$ is also a squared Bessel process.

Theorem

For every $\delta_1, \delta_2 \geq 0$ and $y_1, y_2 \geq 0$ is

$$Q_{y_1}^{\delta_1} \star Q_{y_2}^{\delta_2} = Q_{y_1+y_2}^{\delta_1+\delta_2}$$

$\star$ denotes the convolution.

Ability to pay process

We get for the ability-to-pay-process

$$
\begin{aligned}
X_t^i &= Y_t + Y_t^i \\
&= y_0 + \delta_0 t + 2\int_0^t \sqrt{Y_s}\, dW_s^0 \\
&+ y_0^i + \delta_i t + 2\int_0^t \sqrt{Y_s^i}\, dW_s^i \\
&= (y_0 + y_0^i) + (\delta_0 + \delta_i)t + 2\int_0^t \sqrt{X_s^i}\, d\hat{W}_s
\end{aligned}
$$

Default time

In this setting δ_0 and δ_i have to satisfy the condition $\delta_0 + \delta_i < 2$, since otherwise the default probability of obligor is equal to zero for all $t \geq 0$.
Then the default time of obligor i, defined as

$$T_i = \inf\{t \geq 0 : X_t^i = 0\}$$

is inverse Gamma distributed, i.e.

$$T_i \sim Inv - Gamma\left(1 - \frac{\delta_0 + \delta_i}{2}, \frac{y_0 + y_i}{2}\right)$$

Joint default probability

To determine the joint default probability, i.e. the probability that both defaults occur before time t, it is not possible to exploit the conditional independence of the processes. Under the condition that the path of Y up to time t is known, one would have to determine the probability that the (not necessarily first) hitting times of Y^i and Y^j equal at least one of the hitting times of the common process Y. Thus we have arrived in a dead-end street. But how can we get out?

Ability to pay process

A possible alternative is to substitute the process $(Y_t)_{t\geq 0}$ by $(Y_t^{T_0})_{t\geq 0}$, where Y^{T_0} is a stopped process, i.e. $Y_t^{T_0} = Y_{t\wedge T_0}$. The stopping time T_0 is defined as

$$T_0 = \inf\{t \geq 0 : Y_t = 0\}$$

That means, after the first hitting time of Y, the ability to pay process X consists just of the idiosyncratic part Y^i.

Ability to pay process

Thus, we have

$$\begin{aligned} X_t^i &= \left(y_0^i + y_0 + (\delta_i + \delta_0) \cdot t + 2 \int_0^t \sqrt{X_s^i} d\hat{W}_s \right) \cdot 1_{\{T_0 > t\}} \\ &+ \left(y_0^i + \delta_i \cdot t + 2 \int_0^t \sqrt{Y_s^i} dW_s^i \right) \cdot 1_{\{T_0 \le t\}} \end{aligned}$$

with

$$T_0 = \inf\{t \ge 0 : Y_t = 0\}$$

Default time

The default time of company i is still defined as the first time the process X_t^i reaches zero, i.e.

$$\begin{aligned} T_i &= \inf\{t \geq 0 | X_t^i = 0\} \\ &= \inf\{t \geq 0 | Y_{t \wedge T_0} + Y_t^i = 0\} \\ &= T_0 + \inf\{t \geq 0 | Y_{T_0+t}^i = 0\} \\ &= T_0 + \tilde{T}_i \end{aligned}$$

Default probability

Now we can determine the default probability:

$$
\begin{aligned}
\mathbb{P}(T_i \leq t) &= \mathbb{P}(T_0 + \tilde{T}_i \leq t) \\
&= \mathbb{E}(\mathbb{P}(\tilde{T}_i \leq t - T_0 | T_0)) \\
&= \int_0^\infty \mathbb{P}(\tilde{T}_i \leq t - s)\mathbb{P}(T_0 \in ds) \\
&= \int_0^\infty \mathbb{P}(\tilde{T}_i \leq t - s)1_{\{s \leq t\}}\mathbb{P}(T_0 \in ds) \\
&+ \int_0^\infty \mathbb{P}(\tilde{T}_i \leq t - s)1_{\{s > t\}}\mathbb{P}(T_0 \in ds)
\end{aligned}
$$

Default probability

For the second interval we get

$$
\begin{aligned}
& \int_0^\infty \mathbb{P}(\tilde{T}_i \leq t - s) 1_{\{s>t\}} \mathbb{P}(T_0 \in ds) \\
= & \int_t^\infty \mathbb{P}(\tilde{T}_i \leq t - s) \mathbb{P}(T_0 \in ds) \\
= & \int_0^\infty \mathbb{P}(\tilde{T}_i \leq -s) \mathbb{P}(T_0 \in ds) \\
= & \ 0
\end{aligned}
$$

Default probability

Thus, we have

$$\begin{aligned}
\mathbb{P}(T_i \leq t) &= \int_0^\infty \mathbb{P}(\tilde{T}_i \leq t - s) 1_{\{s \leq t\}} \mathbb{P}(T_0 \in ds) \\
&= \int_0^t \mathbb{P}(\tilde{T}_i \leq t - s) \mathbb{P}(T_0 \in ds) \\
&= \int_0^t \mathbb{P}(\tilde{T}_i \leq t - s) \mathbb{P}(T_0 \in ds) \\
&= \int_0^t \mathbb{P}(\inf\{u | Y^i_{T_0+u} = 0\} \leq t - s) \mathbb{P}(T_0 \in ds)
\end{aligned}$$

Due to the Strong Markov property, conditional on a realization of Y^i, evaluated at the random time T_0, i.e. on the event $\{Y^i_{T_0} = y\}$, the conditional probability distribution of $\tilde{T}_i$ is equal to the distribution of the first hitting time of a Squared Bessel process, starting at y.
Thus

$$\begin{aligned} & \mathbb{P}(\inf\{t \geq 0 : Y^i_{T_0+t} = 0\} \leq s) \\ = \ & \mathbb{E}\left(\mathbb{P}^{\delta_i}_{Y_{T_0}}(\inf\{t \geq 0 : Y^i_t = 0\} \leq s)\right) \end{aligned}$$

Default probability

Now

$$
\begin{aligned}
& \mathbb{E}(\mathbb{P}^{\delta_i}_{Y^i_{T_0}}(\inf\{t|Y^i_t = 0\} \leq s)) \\
= \quad & P_{T_0}\mathbb{P}^{\delta_i}_{y^i_0}(\inf\{t|Y_t = 0\} \leq s) \\
= \quad & \int_0^\infty \mathbb{P}^{\delta_i}_y(\inf\{t|Y_t = 0\} \leq s)p_{T_0}(y^i_0, y)dy
\end{aligned}
$$

where $(P_t)_{t\geq 0}$ is the semi-group of the Squared Bessel process, i.e.

$$
P_t f(x) := \int_0^\infty f(y)p_t(x,y)dy
$$

with the transition probability density $p_t(x, y)$.

Default probability

Then, on the event $\{T_0 = s\}$, we get

$$\int_0^\infty \mathbb{P}_y^{\delta_i}(\inf\{t|Y_t = 0\} \leq s) p_s(y_0^i, y) dy$$

Finally, the probability distribution of the default time T_i evolves into

$$\mathbb{P}(T_i \leq t)$$
$$= \int_0^t \int_0^\infty \mathbb{P}_y^{\delta_i}(\inf\{u|Y_u = 0\} \leq t - s) p_s(y_0^i, y) dy \mathbb{P}(T_0 \in ds)$$

Default probability

$$
\begin{aligned}
& \mathbb{P}_y^{\delta_i}(\inf\{t|Y_t = 0\} \leq s) \\
= \quad & \int_0^s \Gamma\left(1 - \frac{\delta_i}{2}\right)^{-1} \frac{1}{u} \left(\frac{y_0^i}{2u}\right)^{1-\frac{\delta_i}{2}} e^{-\frac{y_0^i}{2u}} du \\
= \quad & F_{IG}(s; 1 - \frac{\delta_i}{2}, \frac{y}{2})
\end{aligned}
$$

where $F_{IG}(x; \alpha, \beta)$ is the probability distribution function of the Inverse Gamma distribution with parameters α and β.

$$
\begin{aligned}
& p_s(y_0^i, y)dy \\
= & \frac{1}{2s}\left(\frac{y}{y_0^i}\right)^{\frac{\delta_i - 1}{4}} \exp\left(-\frac{y_0^i + y}{2s}\right) I_{\frac{\delta_i - 1}{2}}\left(\frac{\sqrt{y_0^i y}}{s}\right) dy \\
= & f_{NC\chi^2}\left(\frac{y}{s}; 1 - \frac{\delta_i}{2}, \frac{y_0^i}{s}\right)\frac{1}{s}dy
\end{aligned}
$$

where $f_{NC\chi^2}(x; k, \lambda)$ is the probability density function of the noncentral χ^2-distribution with k degrees of freedom and noncentrality parameter λ.

Default probability

With these notations, the distribution function of the default time T_i becomes

$$\mathbb{P}(T_i \leq t) = \int_0^t \int_0^\infty F_{IG}\left(t - s; 1 - \frac{\delta_i}{2}, \frac{2}{y}\right) f_{NC\chi^2}\left(\frac{y}{s}; 1 - \frac{\delta_i}{2}, \frac{y_0^i}{s}\right) f_{IG}\left(s; 1 - \frac{\delta}{2}, \frac{2}{y_0}\right) dyds$$

Joint default probability

Now we can easily determine the joint default probability of two firms, that are related via the common process Y_t. Their joint default probability is defined as the probability that both firms have defaulted by time t. Thus, we have

$$
\begin{aligned}
\mathbb{P}(T_1 \leq t, T_2 \leq t) &= \mathbb{P}(T_0 + \tilde{T}_1 \leq t, T_0 + \tilde{T}_2 \leq t) \\
&= \mathbb{E}(\mathbb{P}(\tilde{T}_1 \leq t - T_0, \tilde{T}_2 \leq t - T_0 | T_0)) \\
&= \int_0^\infty \mathbb{P}(\tilde{T}_1 \leq t - s, \tilde{T}_2 \leq t - s) \mathbb{P}(T_0 \in ds) \\
&= \int_0^\infty \mathbb{P}(\tilde{T}_1 \leq t - s) \mathbb{P}(\tilde{T}_2 \leq t - s) \mathbb{P}(T_0 \in ds) \\
&= \int_0^t \mathbb{P}(\tilde{T}_1 \leq t - s) \mathbb{P}(\tilde{T}_2 \leq t - s) \mathbb{P}(T_0 \in ds)
\end{aligned}
$$

Joint default probability

Thus, for the joint default probability we get

$$
\begin{aligned}
& \mathbb{P}(T_1 \leq t, T_2 \leq t) \\
= & \int_0^t \mathbb{P}(\tilde{T}_1 \leq s)\mathbb{P}(\tilde{T}_2 \leq s)\mathbb{P}(T_0 \in ds) \\
= & \int_0^t \int_0^\infty F_{IG}\left(t-s; 1-\frac{\delta_1}{2}, \frac{2}{y_1}\right) f_{NC\chi^2}\left(\frac{y_1}{s}; 1-\frac{\delta_1}{2}, \frac{y_0^1}{s}\right) dy_1 \\
& \int_0^\infty F_{IG}\left(t-s; 1-\frac{\delta_2}{2}, \frac{2}{y_2}\right) f_{NC\chi^2}\left(\frac{y_2}{s}; 1-\frac{\delta_2}{2}, \frac{y_0^2}{s}\right) dy_2 \\
& f_{IG}\left(s; 1-\frac{\delta}{s}, \frac{2}{y_0}\right) ds
\end{aligned}
$$