Pricing Vulnerable Options Using Good Deal Bounds

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Vulnerable Options

- Vulnerable options = options where the writer of the option may default, mainly trading on OTC markets
- BIS, the OTC equity-linked option gross market value in the first half of 2006 USD 6.8 tln

Previous Literature

- Treatment in complete markets (Hull-White(1995), Jarrow-Turnbull(1995), Klein(1996));
- Hung-Liu (2005) : market incompleteness and good deal bound pricing for vulnerable options. Only Wiener process setup.

Contributions of the current paper

- Streamlining the existing literature on vulnerable options in complete markets;
- Applying the Bjork-Slinko (2005) method of computing good deal bounds to obtain higher tractability;
- Applying structural methods for default (intensity based method - work in progress);
- Extending the results for european calls to options with homogeneous payoff functions of the first degree (e.g. exchange options) ;

Pricing in incomplete markets

pricing in incomplete markets $\longrightarrow$ no unique EMM
$\longrightarrow$ no unique price

- classical solutions:
  - no-arbitrage bounds - *too large*
  - choosing one specific martingale measure - *ad-hoc; economic meaning?*
- alternative solution - GOOD DEAL BOUNDS
  - Cochrane and Saa Raquejo (2000)
  - Bjork and Slinko (2005)

Theory of Good Deal Bounds

Main Idea

set a bound on the possible Sharpe Ratio of any portfolio that can be formed on the market $\leftrightarrow$
$\leftrightarrow$ *set a bound on the possible Girsanov kernels for potential EMM*
$\leftrightarrow$ *set a bound on the possible prices for the claim*

Structural Model

specified under the **objective measure P**

- traded stock S

$$dS_t = \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t;$$

- assets of the counterparty Y

$$dY_t = \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t;$$

- bank account B;
- the payoff function of the vulnerable option

$$\Phi(S_T, Y_T) = \max(S_T - K, 0) I(Y_T \geq D) + \mathcal{R} I(Y_T < D);$$

- recovery payoff

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T - K, 0]$$

Good Deal Bound Problem

The **upper good deal bound** price process for a vulnerable option is defined the optimal value process for the following optimal control problem:

$$
\begin{aligned}
\max_{\varphi} \quad & E^{Q}[e^{-r(T-t)}(\max[S_T - K, 0] I\{Y_T \geq D\} + \mathcal{R} I\{Y_T \leq D\})] \\
& dY_t = (\mu_t + \bar{\sigma}_t \varphi_t) Y_t dt + Y_t \bar{\sigma}_t dW_t \\
& dS_t = r S_t dt + S_t \bar{\gamma}_t dW_t \\
& \alpha_t + \bar{\gamma}_t \varphi_t = r \\
& \|\varphi_t\|^2 \leq B^2
\end{aligned}
$$

Hamilton Jacobi Bellman equation

The HJB equation:

$$\frac{\partial V}{\partial t}(t,s,y) + \sup_{\varphi} \mathcal{A}V(t,s,y) - rV(t,s,y) = 0$$
$$V(T,s,y) = \Phi(s,y).$$

is solved in 2 steps:

- solving for each t, s, y the embedded static problem
  $\longrightarrow$ we obtain the Girsanov Kernel
- solving the PDE
  $\longrightarrow$ we obtain the price of the vulnerable option

The Static Embedded Problem

- the static embedded problem

$$\max_{\varphi} \quad \frac{\partial V}{\partial y}\sigma\varphi y$$
$$\alpha + \bar{\gamma}\varphi = r$$
$$\|\varphi\|^2 \leq B^2$$

- the Girsanov kernel

$$\hat{\varphi}'_{U/L} = \left(-\frac{\alpha_t - r}{\gamma_t}, \pm\sqrt{B^2 - \left(\frac{r - \alpha_t}{\gamma_t}\right)^2} \right)$$

Results for Vulnerable European Call

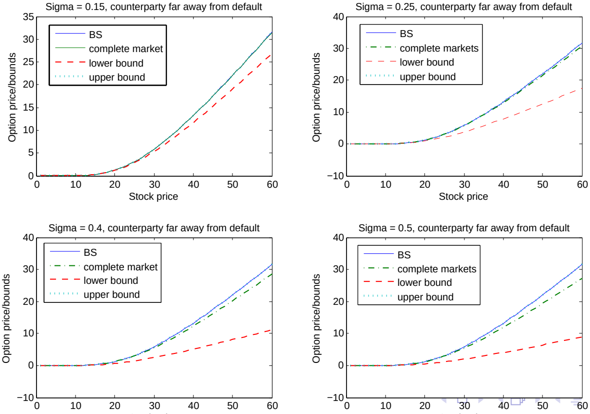
- closed form solution for a vulnerable European call

$$
\begin{aligned}
\Pi(t) &= S_t \mathcal{N}[-a_1, -b_1, \rho] \\
&- e^{-r(T-t)} K \mathcal{N}[-a_2, -b_2, \rho] \\
&+ \frac{1-\beta}{D} S_t Y_t \exp\{\int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \bar{\sigma}_s \bar{\gamma}'_s\right] ds\} \mathcal{N}[-a_3; b_3; -\rho] \\
&- e^{-r(T-t)} \frac{K(1-\beta)}{D} Y_t \exp\{\int_t^T (\mu_s + \bar{\sigma}_s \hat{\varphi}_s) ds\} \mathcal{N}(-a_4, -b_4, \rho)
\end{aligned}
$$

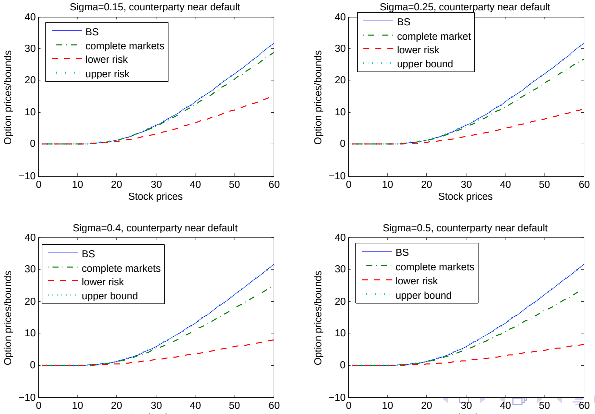
Factors that influence the size of the GDB interval

- factors specific to each transaction
  - distance to default
  - volatility of the assets of the counterparty
  - correlation between the assets of the counterparty and the underlying
  - the size of the market price of risk for the underlying

The Variation of σ. Far from default



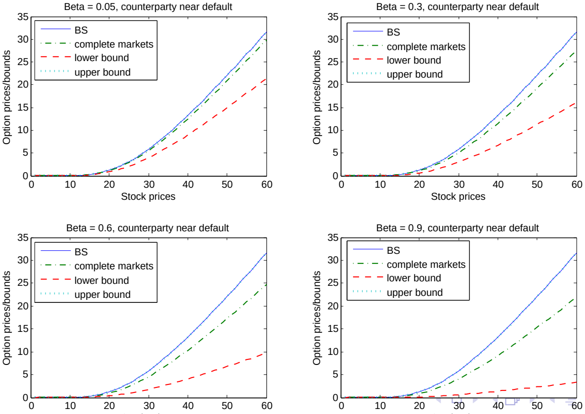
The Variation of σ. Near default



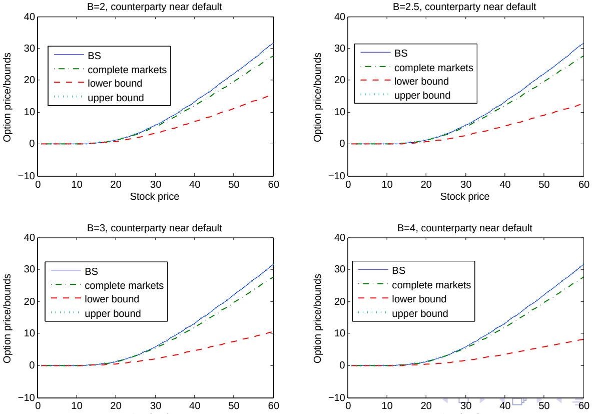
Factors that influence the size of the GDB interval

- factors specific to the market
  - size of the good deal bound constraint (B)
  - the deadweight costs β

The Variation of β. Near default



The Variation of B. Near default



Extensions - exchange options

The payoff of a exchange option

$$\Phi(S_T^1, S_T^2, Y_T, T) = \max[S_T^1 - S_T^2, 0] I\{Y_T \geq D\} + \mathcal{R} I(Y_T < D) \tag{1}$$

- in complete markets, we can price an exchange option by change of measure
- the result extends to vulnerable exchange options
- can we apply the same techniques with GDB?

As in the complete market case:

- having a change of variable for the payoff and martingale conditions;
- re-stating the good deal bound condition:

$$\|\phi\|^2 \leq B^2 \rightarrow \|\psi - \bar{\gamma}_2'\|^2 \leq B^2 \tag{2}$$

- calculating the new relevant Girsanov kernel and correlation coefficient;
- substituting them in the formula for a European call

Barrier Options in Complete markets

payoff

$$C_{LO} = \begin{cases} \max[S_T - K, 0], & \text{if } S_t > L \text{ for all } 0 < t < T \\ 0, & \text{if } S_t \leq L \text{ for some } 0 < t < T \end{cases}$$

remove the path dependency for a vulnerable claim :

$$\begin{aligned} \Pi(0, \Psi^V_{LO}) &= e^{-rT} E^Q_{0,s,y}\left[\Psi^V_L(S_T, Y_T)\right] \\ &- e^{-rT}\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y'}\left[\Psi^V_L(S_T, Y_T)\right] \end{aligned} \quad (3)$$

How?

- We introduce a new process Z_t with the same dynamics as S_t, but starting point $\frac{L^2}{s}$
- Notice that, in this set-up, the payoff of any defaultable claim can be written as:

$$\Psi^V(S_T, Y_T) = \Psi(S_T)F(Y_T),$$

where $F(Y_T) = I\{Y_T \geq D\} + \frac{(1-\beta)Y_T}{D} I\{Y_T < D\}$.

GDB problem for barrier options

The **upper good deal bound** price process for a vulnerable down-and-out option is defined the optimal value process for the following optimal control problem:

$$
\begin{aligned}
\max_{\varphi} \quad & E^{Q}_{0,s,z,y}[e^{-r(T-t)}\Phi(S_T, Z_T)F(Y_T)] \\
& dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t \\
& dS_t = rS_t dt + S_t\bar{\gamma}_t dW_t \\
& dZ_t = rZ_t dt + Z_t\bar{\gamma}_t dW_t \\
& \alpha_t + \bar{\gamma}_t\varphi_t = r \\
& \|\varphi_t\|^2 \le B^2
\end{aligned}
$$

Standard

Conclusion

- We apply the GDB method to vulnerable options;
- We allow for structural models of default;
- We extend the results for European call vulnerable options to other vanilla options with payoff functions homogeneous of the first degree in S.
- We extend results for barrier options when the assets of the counterparty and the underlying are independent

THANK YOU!