On q-Optimal Signed Martingale Measures in Exponential Lévy Models

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Dynamic and Static Problem

Dynamic Problem: $V_\xi(\tilde{x}) = \sup_{Y \in \mathcal{W}_C(\tilde{x})} E[U(Y_T)]$

$$\Updownarrow$$

Static: $\sup_{X \in \Theta^{(2m),\tilde{x}}} E[U(X)] \Leftrightarrow$ Dual: $\sup_{y \geq 0,\ Z \in \mathcal{D}_a^q} E[\check{U}(yZ_T) + \tilde{x}y]$

where

◇ $\mathcal{W}_C(\tilde{x}) = \{Y | Y_t = \tilde{x} + \int_0^t NdS - C_t,\ N \in \mathcal{A}^p,\ C \in \mathcal{K}^p\}$

◇ $\Theta^{(p),\tilde{x}} = \{X \in L^p(\mathcal{F}_T), \forall Z \in \mathcal{D}_a^q\ E(Z_T X) \leq \tilde{x}\},\ \tilde{x} \in \mathbb{R}$.

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Finding Optimal Solutions

dynamic optimization problem

$$\Updownarrow$$

constrained static problem $\longrightarrow$ Lagrange functional $\longrightarrow$ dual problem

optimal $\uparrow$ solution saddle $\uparrow$ point $\downarrow$

$$I(\lambda^*_{\tilde{x}}) := (U')^{-1}(\lambda^*_{\tilde{x}}) \longleftarrow (\lambda^*_{\tilde{x}}, I(Z_{\mathcal{Y}(\tilde{x})}\mathcal{Y}(\tilde{x})) \longleftarrow \lambda^*_{\tilde{x}} = Z_{\mathcal{Y}(\tilde{x})}\mathcal{Y}(\tilde{x})$$

Finding Optimal Solutions

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$$\Updownarrow$$

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$\downarrow$ optimal solution $\uparrow$

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Dual Optimizer via Verification

1. Observe: in many cases, the optimal dual optimizer $\hat{Z}_{\mathcal{Y}(\tilde{x})}$ is independent of $\tilde{x}$ so set $\hat{Z} = \hat{Z}_{\mathcal{Y}(\tilde{x})}$.
2. Propose candidate $\check{Z} \in \mathcal{D}$. ($\mathcal{Y}(\tilde{x})$ can be easily derived if $\hat{Z} = \hat{Z}_{\mathcal{Y}(\tilde{x})}$).
3. $X_0(\tilde{x}) := I(\mathcal{Y}(\tilde{x})\check{Z}_T)$ is optimal solution of $\sup_X E(U(X))$, s.t. $E(\check{Z}_T X) \leq \tilde{x}$
4. Find strategy to replicate $X_0(\tilde{x})$
   $\Rightarrow X_0(\tilde{x})$ is optimal terminal value of the dynamic problem

$\Rightarrow$ $X_0(\tilde{x})$ is optimal solution of $\sup_X E(U(X))$, s.t. $\forall Z \in \mathcal{D}: \ E(Z_T X) \leq \tilde{x}$

$\Rightarrow$ By the duality relation $\mathcal{Y}(\tilde{x})\hat{Z}_T := \mathcal{Y}(\tilde{x})\check{Z}_T$ is optimal dual solution.

Convergence to the primal and dual solution of the exponential problem

$$
\begin{array}{ccc}
\mathcal{Y}_{2m}(\tilde{x})Z_{2m} & \longrightarrow & \mathcal{Y}_{exp}(\tilde{x})Z_{min} \\
\cong & & \cong \\
X_0^{(2m)}(\tilde{x}) & \longrightarrow & X_0^{(exp)}(\tilde{x}) \\
\cong & & \cong \\
V_{2m}(\tilde{x}) & \longrightarrow & V_{exp}(\tilde{x}) \\
\cong & & \cong \\
\phi_{2m}(\mathcal{Y}_{2m}(\tilde{x})) & \longrightarrow & \phi_{exp}(\mathcal{Y}_{exp}(\tilde{x})) \\
\cong & & \cong \\
\vartheta^{(2m)} & \longrightarrow & \vartheta^{exp}
\end{array}
$$

(convergence of dual solutions)

(convergence of terminal wealths)

(convergence of value functions)

(convergence of the dual functions)

(convergence of portfolios)

Market Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in (0, \infty)$ a finite time horizon, and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration satisfying the usual conditions, i.e. right-continuity and completeness.

We suppose that a discounted market with n assets is given by

$$S_t = \operatorname{diag}(S_0^{(1)}, \dots, S_0^{(n)})e^{\check{X}_t}, \quad t \in [0, T],\ S_0^{(i)} > 0, \tag{1}$$

where $\check{X}$ is supposed to be an $\mathbb{R}^n$-valued Lévy process with characteristic triplet $(\sigma\sigma^*, \nu, b)$ on $(\Omega, \mathcal{F}, P)$ and N a Poisson random measure with intensity measure $\nu(dx)dt$, and $\tilde{N}(dx, dt) = N(dx, dt) - \nu(dx)dt$ appearing in the Lévy-Itô-decomposition.

Market Model: Semimartingale Decomposition

We assume that the filtration $\mathbb{F}$ coincides with $\mathbb{F}^{\check{X}}$, the completion of the filtration generated by the Lévy process $\check{X}$ and $E[|S(t)|] < \infty$ for all $t \in [0, T]$. The second assumption guarantees that S is a special semimartingale with decomposition $S_t = S_0 + M_t + A_t$, where

$$dM_t = \mathbf{S}_{t-}(\sigma dW_t + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1})\tilde{N}(dx, dt))$$

and

$$dA_t = \mathbf{S}_{t-}(-\beta + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1} - x 1_{\|x\| \leq 1})\nu(dx))dt.$$

Here, $\beta = -(b + \frac{1}{2}\sum_j \sigma_{\cdot,j}^2)$ and $\mathbf{S} = diag(S^{(1)}, ..., S^{(n)})$. $\mathbf{1}$ denotes the vector in $\mathbb{R}^n$ having all entries equal to one, and expressions such as e^x are to be interpreted componentwise, i.e. $e^x = (e^{x_1}, ..., e^{x_n})'$.

Minimal Entropy Martingale Measure

We set,

$$\mathcal{D}_s^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ SZ \text{ is a local } P\text{-martingale}\},$$

where $\mathcal{U}^q$ denotes the set of $\mathbb{R}$-valued $L^q(\Omega, P)$-uniformly integrable martingales. A subset is $\mathcal{D}_a^q = \{Z \in \mathcal{D}_s^q\ Z_T \geq 0\ P\text{-a.s.}\}$. We recall the definition of the minimal entropy martingale measure:

(Min$_{e,\log}$) Find $Z^{\min} \in \mathcal{D}_a^{\log}$ such that

$$E[Z_T^{\min} \log Z_T^{\min}] = \inf_{Z \in \mathcal{D}_a^{\log}} E[Z_T \log Z_T].$$

where

$$\mathcal{D}_a^{\log} = \{Z \in \mathcal{D}_a^1,\ E(Z_T \log Z_T) < \infty\}.$$

$dQ_{\min} = Z_T^{\min} dP$ is called the *minimal entropy martingale measure* $Q_{\min}$ (MEMM).

q-Optimal Martingale Measures

We further study the q-optimal signed martingale measure:

($\mathbf{Min}_{s,q}$) Find $Z^{(q)} \in \mathcal{D}_s^q$ such that

$$E[|Z_T^{(q)}|^q] = \inf_{Z \in \mathcal{D}_s^q} E[|Z_T|^q].$$

$dQ^{(S,q)} = Z_T^{(q)} dP$ is called the *q-optimal signed martingale measure* $Q^{(S,q)}$ (qSMM).

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Replace, $\mathcal{D}_s^q$ by $\mathcal{D}_e^q = \{Z \in \mathcal{D}_s^q \; Z_T > 0 \; P\text{-a.s.}\}$, then the solution, provided its existence, defines the *q-optimal equivalent martingale measure* $Q^{(e,q)}$ (qEMM) with density process $\tilde{Z}^{(q)}$.

Assumptions

Assumption (C_q)

$\mathbf{C_q^-}$: There exists an $\theta_q \in \mathbb{R}^n$ such that

$$\mathrm{eg}_q(x) := ((q-1)\theta_q'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}}$$

defines a real-valued function on the support on ν which satisfies

$$\sigma\sigma'\theta_q + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1})\mathrm{eg}_q(x) - x1_{\|x\|\leq 1}\nu(dx) = \beta \tag{2}$$

and

$$\int_{\mathbb{R}_0^n} |(\mathrm{eg}_q^q(x)) - 1 - q(\mathrm{eg}_q(x) - 1)|\nu(dx) < \infty. \tag{3}$$

$\mathbf{C_q^+}$: $(q-1)\theta_q'(e^y - \mathbf{1}) + 1 > 0, \nu$-a.s.

If C_q^- and C_q^+ are satisfied, we say that C_q holds.

q-Optimal Equivalent Martingale Measure: Existence

Theorem (Jeanblanc et al., Theorem 2.9)

Suppose C_q holds. Then the qEMM exists and is given by

$$\mathcal{E}(\theta_q' \sigma, \mathrm{eg}_q - 1),$$

where $\mathcal{E}(f, g)$ denotes the stochastic exponential with Girsanov parameters f, g, i.e.

$$\begin{aligned}\mathcal{E}_t(f, g) &= e^{\int_0^t f(s) dW_s - \frac{1}{2} \int_0^t \|f(s)\|^2 ds + \int_0^t \int_{\mathbb{R}_0^n} g(s,x) \tilde{N}(dx,ds)} \\ &\quad \times \prod_{s \leq t} (1 + g(s, \Delta \check{X}(s))) e^{-g(s, \Delta \check{X}(s))}.\end{aligned}$$

However, C_q^+ is very restrictive!

q-Optimal Equivalent Martingale Measure: Problems

Proposition

Suppose $n = 1$ and P is not a martingale measure. Then:
(i) If C_q holds for some $q > 1$, then

$$\int_{x \geq 1} e^{\theta e^x} \nu(dx) < \infty \tag{4}$$

for some $\theta > 0$ or the minimal entropy martingale measure does not exist.
(ii) If C_q holds for some $q > 1$, then

$$\int_{\mathbb{R}_0} (e^x - 1) - x 1_{|x| \leq 1} \nu(dx) + (b + \frac{1}{2}\sigma^2) < 0 \tag{5}$$

or upward jumps are bounded, i.e. $\nu([L, \infty)) = 0$ for some $L > 0$.

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q-Optimal Equivalent Martingale Measure: Problems

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or upward jumps are bounded, i.e. $\nu([L, \infty)) = 0$ for some $L > 0$.

In particular, in a Kou or a Merton model C_q and the existence of the minimal entropy martingale measure cannot hold simultaneously. Condition (5) is rather unlikely (a negative optimal portfolio is induced). ▸ Examples

q-Optimal Signed Martingale Measure: Existence

Theorem

Suppose that $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$ and that C_q^- holds. Then,

$$Z^{(q)} = \mathcal{E}(\theta_q' \sigma, \mathrm{eg}_q - 1)$$

is the density process of qSMM.

Proposition

Suppose $n = 1$, $q(m) = \frac{2m}{2m-1}$, P is not a martingale measure, and the set of equivalent martingale measures is nonempty. Then, $C_{q(m)}^-$ holds for $m \in \mathbb{N}$, if and only if

$$\int_{x \geq 1} e^{2mx} \nu(dx) < \infty. \tag{6}$$

q-Optimal Signed Martingale Measure: Examples

Example

Suppose $n = 1$.
(i) If $\nu(dx)$ behaves (up to a slowly varying function) as $e^{-\lambda_+ x}dx$ for $x \to \infty$, then $C^-_{q(m)}$ holds for $m < \lambda_+/2$ and fails for $m > \lambda_+/2$. However, C_q fails for all q, if $\int_{\mathbb{R}_0}(e^x - 1) - x1_{|x|\leq 1}\nu(dx) + (b + \frac{1}{2}\sigma^2) > 0$. This tail behavior is inherent in generalized hyperbolic models and the Kou model.
(ii) If there are constants $\eta_0, \eta_1 > 0$ such that

$$\int_{x\geq 1} e^{\eta_0 x^{1+\eta_1}}\nu(dx) < \infty, \tag{7}$$

then $C^-_{q(m)}$ holds for all $m \in \mathbb{N}$. However C_q fails for all q, if the upward jumps are not bounded and $\int_{\mathbb{R}_0}(e^x - 1) - x1_{|x|\leq 1}\nu(dx) + (b + \frac{1}{2}\sigma^2) > 0$.
A popular model, which satisfies (7) and has unbounded upward jumps is the Merton model.

Minimal Entropy Martingale Measure

Assumption (C)

There exists a vector $\theta_e \in \mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}_0^n} \|(e^x - \mathbf{1})e^{\theta_e'(e^x - \mathbf{1})} - x1_{\|x\|\leq 1}\|\nu(dx) < \infty \tag{8}$$

and $\theta_e'\sigma\sigma' + \int_{\mathbb{R}_0^n}(e^x - \mathbf{1})e^{\theta_e'(e^x-\mathbf{1})} - x1_{\|x\|\leq 1}\nu(dx) = \beta$.

Theorem (Fujiwara/Miyahara or Esche/Schweizer and Hubalek/Sgarra)

(i) If condition C is satisfied, then the entropy minimal martingale measure is given by

$$\mathcal{E}(\theta_e'\sigma, e^{\theta_e(e^x-1)} - 1).$$

(ii) If $n = 1$ and there is no θ_e satisfying C, then the entropy minimal martingale measure does not exist.

Convergence to the Minimal Entropy Martingale Measure

Theorem (MEMM)

Suppose $n = 1$, the minimal entropy martingale measure exists, and there is a $\delta > 0$ such that θ_e, specified by condition C, satisfies

$$\int_{x \geq 1} e^{(\max\{\theta_e, -0.28\,\theta_e\}+\delta)e^x} \nu(dx) < \infty. \tag{9}$$

Then:

(i) If $\theta_e > 0$ or upwards jumps are bounded, then C_q is satisfied for sufficiently small $q > 1$ and the q-optimal equivalent martingale measures converge to the minimal entropy martingale measure in $L^r(P)$, for some $r > 1$, as $q \downarrow 1$ (in the sense that the densities converge).

(ii) Suppose $q(m) = \frac{2m}{2m-1}$. If $\theta_e < 0$, then $C^-_{q(m)}$ is satisfied for all $m \in \mathbb{N}$ and the $q(m)$-optimal signed martingale measures converge to the minimal entropy martingale measure in $L^r(P)$, for some $r > 1$, as $m \uparrow \infty$.

Verification in a General Semimartingale Model

Theorem

Suppose $\hat{Z} \in \mathcal{D}_s^q$, $q = \frac{2m}{2m-1}$ and, for some $\tilde{x} < 2m$, the contingent claim

$$X^{(2m)}(\hat{Z}) := 2m - 2m\hat{Z}_T^{\frac{1}{2m-1}} \left(\frac{2m - \tilde{x}}{2mE(\hat{Z}_T^{\frac{2m}{2m-1}})} \right) \tag{10}$$

is replicable with a predictable strategy ϑ (#shares held) and $\vartheta \in \mathcal{A}^{2m}$, i.e.

$$\|\vartheta\|_{L^{2m}(M)} := \|(\int_0^T \vartheta d[M]_t \vartheta')^{\frac{1}{2}}\|_{L^{2m}(\Omega,P)} < \infty, \tag{11}$$

$$\|\vartheta\|_{L^{2m}(A)} := \| \int_0^T |\vartheta dA_t| \|_{L^{2m}(\Omega,P)} < \infty. \tag{12}$$

Then $\hat{Z}$ is the density process of the q-optimal signed martingale measure.

▸ Proof

Replicating Strategy in the above Lévy Setting

Lemma

Suppose that $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$ and that C_q^- holds. Define

$$\vartheta_t^{(2m)} = -\frac{2m-\tilde{x}}{2m-1}\mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1})) \times \theta_q' \mathbf{S}_{t-}^{-1} e^{t(q-1)\theta_q'\left(-\beta + \int_{\mathbb{R}_0^n}(e^x - \mathbf{1} - x 1_{\|x\| \le 1})\nu(dx)\right)}.$$

Then for $\tilde{x} \le 2m$ and $\hat{Z} = \mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1)$ the contingent claim

$$X^{(2m)}(\hat{Z}) := 2m - 2mZ_q^{\frac{1}{2m-1}} \left(\frac{2m-\tilde{x}}{2mE(\hat{Z}_T^{\frac{2m}{2m-1}})}\right)$$

is replicable with initial wealth $\tilde{x}$ and the predictable strategy $\vartheta^{(2m)} \in \mathcal{A}^{2m}$.

Consequences for Portfolio Management

The replicating strategy $\vartheta^{(2m)}$ is the solution of

$$\arg\max\{E(u_{2m}(X));\ X \in \Theta^{(2m),\tilde{x}}\} \tag{13}$$

with respect to the utility function $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$, where

$$\Theta^{(2m),\tilde{x}} = \left\{X \in L^{2m}(\Omega, \mathcal{F}_T, P) :\ \exists \vartheta \in \mathcal{A}^{(2m)} \text{ s.t. } X = \tilde{x} + \int_0^T \vartheta_u dS_u\right\}.$$

Moreover under the assumptions of Theorem MEMM, $\vartheta^{(2m)}$ converges uniformly in probability to the optimal portfolio of the exponential problem, $U(x) = -e^{-x}$, $\vartheta^{(\infty)}$ and

$$\lim_{m\to\infty} \sup_{0\le t\le T} \left|(x + \int_0^t \vartheta_u^{(2m)} dS_u) - (x + \int_0^t \vartheta_u^{(\infty)} dS_u)\right|$$

Note, if S is a one-dimensional (non-compensated) exponential Poisson process with jump height 2, there will be arbitrage but the portfolio problem (13) has a solution with $\theta_{q(m)} = -(2m-1)$!

Conclusion

1. in the presence of jumps the q-optimal measure may fail to be equivalent, but belongs to the larger class of signed martingale measures
2. an analogous representation for the densities of equivalent martingale measures as stochastic exponentials is not available
   $\Rightarrow$ techniques for the equivalent case cannot be generalized
3. a verification procedure based on a hedging problem yields an explicit representation of the q-optimal signed martingale measure
4. restrictive conditions for the equivalent case can be dropped
   $\Rightarrow$ in many practically relevant models qMMM is signed
5. necessary and sufficient conditions for the existence of the q-optimal signed and equivalent measure are presented
6. convergence of the q-optimal measures to the minimal entropy martingale measure is established
7. consequences for the exponential utility problem are discussed

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q-Optimal Equivalent Martingale Measure: Examples

back

(i) Note that most of the concrete models discussed in the literature, such as generalized hyperbolic models or the popular jump-diffusion models by Merton or Kou satisfy $\int_{x\geq 1} e^{\theta e^x}\nu(dx) = \infty$ for all $\theta > 0$. Hence, C_q and the existence of the MEMM cannot hold simultaneously for these models.

(ii) In condition (5) upward jumps are exponentially weighted and downward jumps are exponentially damped. Hence,

$$\int (e^x - 1) - x1_{|x|\leq 1}\nu(dx)$$

can become negative only, if the Lévy measure gives much more weight to negative jumps than to positive jumps, leading to an extreme gain-loss asymmetry in the jumps. In such situation we expect that the deterministic trend b is large to compensate for the risk of downward jumps. So condition (5) may be rather unlikely to occur.

Proof

▸ Verification

Let $q = 2m/(2m-1)$. We consider the following maximization problems with utility function $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$:

$$\begin{aligned} \text{Max}_1: \ X^{(1)} &:= \arg\max\{E(u_{2m}(X));\ X \text{ s.t. } E(\hat{Z}_T X) \leq \tilde{x}\} \\ \text{Max}_2: \ X^{(2)} &:= \arg\max\{E(u_{2m}(X));\ X \text{ s.t. } \forall Z \in \mathcal{D}_s^q:\ E(Z_T X) \leq \tilde{x}\} \\ \text{Max}_3: \ X^{(3)} &:= \arg\max\{E(u_{2m}(X));\ X \in \Theta^{(2m),\tilde{x}}\} \end{aligned}$$

where

$$\Theta^{(2m),\tilde{x}} = \left\{ X \in L^{2m}(\Omega, \mathcal{F}_T, P):\ \exists \vartheta \in \mathcal{A}^{(2m)} \text{ s.t. } X = \tilde{x} + \int_0^T \vartheta_u dS_u \right\}.$$

Proof

▸ Verification

We have

$$E(u_{2m}(X^{(1)})) \geq E(u_{2m}(X^{(2)})) \geq E(u_{2m}(X^{(3)})) \geq E(u_{2m}(X^{(2m)}(\hat{Z}))).$$

A straightforward calculation shows that the convex dual of u_{2m} is given by

$$\check{u}_{2m}(y) = (2m-1)y^{2m/(2m-1)} - 2my.$$

Standard duality theory can be applied to verify that $X^{(2m)}(\hat{Z})$ is the maximizer of problem Max_1. All inequalities turn into identities. Moreover,

$$E(u_{2m}(X(\hat{Z}))) = E(u_{2m}(X^{(2)})) \leq \inf_{Z\in\mathcal{D}_s^q,\, y\geq 0} (E(\check{u}_{2m}(y \cdot Z_T)) + \tilde{x}y)$$

$$= \inf_{y\geq 0} \left((2m-1)y^{2m/(2m-1)} \left(\inf_{Z\in\mathcal{D}_s^q} E[Z_T^{2m/(2m-1)}] \right) - (2m-\tilde{x})y \right)$$

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