Dynamic convex risk measures: time consistency, prudence, and sustainability

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Risk Measures and dynamical setting

- Axiomatic analysis of capital requirements:

  A risk measure ρ is a map $L^\infty \to \mathbb{R}$ satisfying certain axioms.
  (Artzner et al. (1997, 1999), Delbaen (2000), Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002))

- Dynamical setting: Filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, P), \quad \mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F} = \mathcal{F}_T.$$

  The time horizon T might be finite or infinite.

  The risk of a financial position $X \in L^\infty(\mathcal{F}_T)$ is evaluated by the risk process $(\rho_t(X))_{t=0,1,\dots}$.
  ρ_t is a map $L^\infty \to L^\infty(\mathcal{F}_t)$ taking into account information $\mathcal{F}_t$.
  (Artzner et al. (2004), Cheridito et al. (2006), Delbaen (2003), Detlefsen (2003), Scandolo (2003), Riedel (2004))

Conditional Convex Risk Measure

A map $\rho_t : L^\infty \to L^\infty(\mathcal{F}_t)$ with the following properties for all $X, Y \in L^\infty$:

- Conditional Translation Invariance: $\forall\, X_t \in L_t^\infty$:

$$\rho_t(X + X_t) = \rho_t(X) - X_t$$

- (Inverse) Monotonicity: $X \leq Y \;\; \Rightarrow \;\; \rho_t(X) \geq \rho_t(Y)$
- Conditional Convexity: $\forall\, \lambda \in L_t^\infty,\ 0 \leq \lambda \leq 1$:

$$\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda\rho_t(X) + (1-\lambda)\rho_t(Y)$$

- Normalization: $\rho_t(0) = 0$.

is called a conditional convex risk measure.

$\phi_t := -\rho_t$ is a conditional monetary utility function.

Acceptance Sets

An important characterization of a conditional convex risk measure is the acceptance set

$$\mathcal{A}_t := \left\{ X \in L^\infty \,\middle|\, \rho_t(X) \leq 0 \right\}.$$

ρ_t is uniquely determined through its acceptance set:

$$\rho_t(X) = \operatorname{ess\,inf} \left\{ Y \in L^\infty_t \,\middle|\, X + Y \in \mathcal{A}_t \right\}.$$

$\longrightarrow$ $\rho_t(X)$ is the minimal conditional capital requirement needed to make a financial position X acceptable at time t.

Robust Representation (*cf. Detlefsen and Scandolo (2005)*)

For a conditional convex risk measure ρ_t the following are equivalent:

- ρ_t is continuous from above;
- ρ_t has the robust representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} \left(E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q) \right),$$

where the penalty function $\alpha_t^{\min}$ is given by

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X \in L^\infty} \left(E_Q[-X \,|\, \mathcal{F}_t\,] - \rho_t(X) \right) = \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X \,|\, \mathcal{F}_t\,]$$

for $Q \in \mathcal{P}_t := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q \approx P \text{ on } \mathcal{F}_t \right\}$.

Problem

In the dynamical setting we obtain for each X a sequence of risk assessments $(\rho_t(X))_{t=0,1,\ldots}$. The question arises:

How are the risk assessments at different times interrelated?

$\longrightarrow$ Several notions of **time consistency**.

(Strong) Time Consistency

- A dynamic convex risk measure $(\rho_t)_{t=0,1,\dots}$ is called (strongly) time consistent, if for all $X, Y \in L^\infty$ and $t \geq 0$ the following holds:

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \quad \Rightarrow \quad \rho_t(X) = \rho_t(Y).$$

Equivalent characterization of (strong) time consistency is

- Recursiveness:

$$\rho_t = \rho_t(-\rho_{t+1}) \qquad \forall\, t \geq 0.$$

(Artzner et al. (2004), Cheridito et al. (2006), Delbaen (2003), Detlefsen and Scandolo (2005), Klöppel and Schweizer (2005), Riedel (2004))

Step by Step

Consider a conditional convex risk measure ρ_t restricted to the space $L^\infty(\mathcal{F}_{t+1})$, i.e. just looking one step ahead.

The corresponding "one-step" acceptance set is given by

$$\mathcal{A}_{t,t+1} := \{\, X \in L^\infty(\mathcal{F}_{t+1}) \mid \rho_t(X) \le 0 \}$$

and the minimal "one-step" penalty function by

$$\alpha^{\min}_{t,t+1}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_{t,t+1}} E_Q[-X \mid \mathcal{F}_t\,], \quad Q \in \mathcal{P}_t.$$

Equivalent Characterizations

Let $(\rho_t)_{t=0,1,\dots}$ be a dynamic convex risk measure such that each ρ_t is continuous from above and assume that the set

$$\mathcal{Q}^* := \left\{ Q \in \mathcal{M}^e(P) \,\middle|\, \alpha_0^{\min}(Q) < \infty \right\}$$

is nonempty. Then the following conditions are equivalent:

1. $(\rho_t)_{t=0,1,\dots}$ is (strongly) time consistent.
2. $\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \qquad \forall\, t.$
3. $\alpha_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \qquad \forall\, t,\ \forall\, Q \in \mathcal{M}^e(P).$
4. $(\rho_t(X) + \alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale $\quad \forall\, Q \in \mathcal{Q}^*$.

In each case ρ_t has a robust representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right), \quad t = 0, 1, \dots .$$

Dynamics of Penalty Functions

In particular it follows that the penalty function process $(\alpha_t^{\min}(Q))$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$ with the Riesz decomposition

$$\alpha_t^{\min}(Q) = \underbrace{E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\middle|\, \mathcal{F}_t\right]}_{Q\text{-potential}} + \underbrace{\lim_{s\to\infty} E_Q\left[\alpha_s^{\min}(Q)|\mathcal{F}_t\right]}_{Q\text{-martingale}}$$

and the Doob-decomposition

$$\alpha_t^{\min}(Q) = E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\middle|\, \mathcal{F}_t\right] + M_t^Q - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q).$$

Prudence

We introduce weaker notion of time consistency:

- A dynamic convex risk measure $(\rho_t)_{t=0,1,\dots}$ is called prudent, if

$$\rho_t \geq \rho_t(-\rho_{t+1}) \qquad \forall\, t \geq 0$$

or equivalently

$$\rho_t(\underbrace{\rho_t(X) - \rho_{t+1}(X)}_{\text{adjustment at } t+1}) \leq 0 \qquad \forall\, t \geq 0, \forall\, X.$$

- Another equivalent characterization of prudence is

$$X \in \mathcal{A}_t \quad \Rightarrow \quad -\rho_{t+1}(X) \in \mathcal{A}_t \qquad \forall\, t \geq 0, \forall\, X$$

("stay on the safe side").

Equivalent Characterizations

Let $(\rho_t)_{t=0,1,...}$ be a dynamic convex risk measure such that each ρ_t is continuous from above and sensitive. Then the following conditions are equivalent:

1. $(\rho_t)_{t=0,1,...}$ is prudent.
2. $\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$ for all t.
3. $\alpha_t^{\min}(Q) \leq \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q)|\mathcal{F}_t]$ for all t and all $Q \in \mathcal{M}^e(P)$.

Equivalent Characterizations (continued)

Moreover, properties 1) - 3) imply the following:

4. The process

$$\rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \ldots$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}_{\infty,\mathrm{loc}}$, where

$$\mathcal{Q}_{\infty,\mathrm{loc}} := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; E_Q\left[\textstyle\sum_{k=0}^{t} \alpha_{k,k+1}^{\min}(Q)\right] < \infty \;\forall\, t \geq 0 \right\}.$$

Assume further that either $T < \infty$ or

$$\exists\, Q^* \in \mathcal{M}^e(P) \;\text{ such that }\; \alpha_{t,t+1}^{\min}(Q^*) \in L^\infty(\mathcal{F}_t) \;\forall\, t \geq 0.$$

Then property 4) is equivalent to properties 1) - 3).

Sustainability

Let $(\rho_t)_{t=0,1,\ldots}$ be a dynamic risk measure and let $X = (X_t)_{t=0,1,\ldots}$ be a bounded adapted process. Then we call X sustainable with respect to the risk measure (ρ_t), if

$$\rho_t(X_t - X_{t+1}) \leq 0 \qquad \text{for all } t = 0, 1, \ldots.$$

- **Meaning**: We consider X to be a cumulative investment process. Then $X_{t+1} - X_t$ is an adjustment that has to be added at time $t + 1$. If the process X is sustainable, then this future payment is acceptable with respect to the risk measure ρ_t.
- A dynamic risk measure (ρ_t) is prudent iff for each X the risk process $(\rho_t(X))$ is sustainable with respect to (ρ_t).

Sustainability (continued)

Suppose that $(\rho_t)_{t=0,1,\dots}$ is a dynamic convex risk measure such that each ρ_t is continuous from above and let X be any bounded adapted process. Consider the following properties:

a) The process X is sustainable with respect to the risk measure (ρ_t).

b) The process

$$X_t - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q), \qquad t = 0, 1, \dots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}_{\infty,\mathrm{loc}}$.

Then property a) implies property b). Assume further that

$$\exists\, Q^* \in \mathcal{M}^e(P) \text{ such that } \alpha^{\min}_{t,t+1}(Q^*) \in L^\infty(\mathcal{F}_t)\ \forall\, t \geq 0.$$

Then properties a) and b) are equivalent.

Recursive construction

Suppose that $T < \infty$ and let $(\rho_t)_{t=0,\ldots,T}$ be a dynamic convex risk measure. Consider a new risk measure $(\widetilde{\rho}_t)_{t=0,\ldots,T}$ defined recursively by

$$\begin{aligned} \widetilde{\rho}_T(X) &:= \rho_T(X) = -X \\ \widetilde{\rho}_t(X) &:= \rho_t(-\widetilde{\rho}_{t+1}(X)), \qquad t = 0, \ldots, T-1, \;\; X \in L^\infty. \end{aligned}$$

- Then $(\widetilde{\rho}_t)$ is again a dynamic convex risk measure and it is (strongly) time consistent by definition. *(cf. Cheridito et al. (2006), Cheridito and Kupper (2006), Drapeau (2006))*
- If the original risk measure (ρ_t) is prudent, then $(\widetilde{\rho}_t)$ lies below (ρ_t).

Recursive construction (continued)

Suppose that $T < \infty$ and let $(\rho_t)_{t=0,\dots,T}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Assume further that for each $t = 1, \dots, T-1$

$$\exists\, Q_t \in \mathcal{M}_1(P) \ : \ Q_t \approx P \text{ on } \mathcal{F}_{t+1},\ E_{Q_t}\left[\alpha^{\min}_{t,t+1}(Q_t)\right] < \infty.$$

Let $(\widetilde{\rho}_t)_{t=0,\dots,T}$ denote the (strongly) time consistent dynamic convex risk measure that arises from (ρ_t) via recursive construction.

Then for each $X \in L^\infty$ the risk process $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ is the smallest bounded adapted process such that it is sustainable with respect to (ρ_t) and covers the final loss.

Literature

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