Optimal Investment under Dynamic Risk Constraints and Partial Information

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Outline

Model Setup

- **Filtered probability space**: $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$
- **Finite time horizon:** $T > 0$
- **Money market**: bond with stochastic interest rates r

$$\mathrm{d}S_t^{(0)} = S_t^{(0)} r_t \,\mathrm{d}t\,, \quad S_0^{(0)} = 1\,, \quad \text{i.e.,} \quad S_t^{(0)} = \exp\Big(\int_0^t r_s \,\mathrm{d}s\Big)\,,$$

  r uniformly bounded and progressively measurable w.r.t. $\mathcal{F}$

- **Stock market**: n stocks with price process $S_t = (S_t^{(1)}, \dots, S_t^{(n)})^\top$, return R_t, and excess return $\tilde{R}_t$, where

$$\mathrm{d}S_t = \mathrm{Diag}(S_t)(\mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t)\,, \quad \mathrm{d}R_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t\,, \quad \mathrm{d}\tilde{R}_t = \mathrm{d}R_t - r_t \,\mathrm{d}t\,.$$

  W n-dimensional standard Brownian motion w.r.t. $\mathcal{F}$ and P
  drift $\mu_t \in \mathbb{R}^n$ $\mathcal{F}_t$-adapted and independent of W
  volatility $\sigma_t \in \mathbb{R}^{n\times n}$ progressively measurable w.r.t. $\mathcal{F}_t^S$,
  σ_t non-singular, and σ_t^{-1} uniformly bounded.

Risk Neutral Probability Measure

We introduce the risk neutral probability measure ($\rightarrow$ for filtering and optimization).

Definition

- Martingale density process

$$Z_t = \exp\left(- \int_0^t \theta_s^\top \, \mathrm{d}W_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 \, \mathrm{d}s \right)$$

  with $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_n)$ the market price of risk

- Risk neutral probability measure $\tilde{\mathrm{P}}$ defined by

$$\frac{\mathrm{d}\tilde{\mathrm{P}}}{\mathrm{d}\mathrm{P}} := Z_T$$

  $\tilde{\mathrm{E}}$ expectation operator under $\tilde{\mathrm{P}}$

- Girsanov's theorem:

$$\tilde{W}_t := W_t + \int_0^t \theta_s \, \mathrm{d}s$$

  defines a $\tilde{\mathrm{P}}$-Brownian motion

Partial Information

Remark

- We consider the case of partial information:
  $\rightarrow$ we can only observe interest rates and stock prices ($\mathcal{F}^{r,S}$) but **not the drift**
- The portfolio has to be adapted to $\mathcal{F}^{r,S}$
  $\rightarrow$ we need the conditional density $\zeta_t = \mathrm{E}[Z_t|\mathcal{F}_t^S]$
  $\rightarrow$ we need the filter for the drift $\hat{\mu}_t = \mathrm{E}[\mu_t|\mathcal{F}_t^S]$

Assumption

- The interest rates r are $\mathcal{F}^S$-adapted $\rightarrow \mathcal{F}^{r,S} = \mathcal{F}^S$
- Z is a martingale w.r.t. $\mathcal{F}$ and P

Lemma

- *We have* $\mathcal{F}^S = \mathcal{F}^{\tilde{W}} = \mathcal{F}^{\tilde{R}}$ $\rightarrow$ *the market is complete w.r.t.* $\mathcal{F}^S$

Outline

Consumption and Trading Strategy

Definition

- Trading strategy π_t: n-dimensional, $\mathcal{F}^S$-adapted, measurable
- Initial capital $x_0 > 0$
- Wealth process X^π satisfies

$$
\begin{aligned}
\mathrm{d}X_t^\pi &= \pi_t^\top(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t) + (X_t^\pi - \mathbf{1}_n^\top \pi_t) r_t\,\mathrm{d}t \\
X_0^\pi &= x_0
\end{aligned}
$$

- A strategy is admissible if $X_t^\pi \geq 0$ a.s. for all $t \in [0, T]$

π_t represents the wealth invested in the stocks at time t

$\eta_t^\pi = \pi_t / X_t^\pi$ denotes the corresponding fraction of wealth

Utility Functions

Definition

$U\colon [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ is a utility function, if U is strictly increasing, strictly concave, twice continuously differentiable on $(0,\infty)$, and satisfies the Inada conditions:

$$U'(\infty) = \lim_{x\to\infty} U'(x) = 0\ , \qquad U'(0+) = \lim_{x\downarrow 0} U'(x) = \infty\ .$$

I denotes the inverse function of U'.

Assumption

$I(y) \le Ky^a, \quad |I'(y)| \le Ky^{-b}$ for all $y \in (0,\infty)$ and $a, b, K > 0$

Example

Logarithmic utility $U(x) = \log(x)$ Power utility $U(x) = x^\alpha/\alpha$ for $\alpha < 1, \alpha \neq 0$.

Optimization Problem

Optimization Problem

We optimize under partial information!

Objective: Maximize the expected utility from terminal wealth, i.e.,

$$\text{maximize} \;\; \mathrm{E}\big[U(X_T)\big]$$

under (risk) constraints we still have to specify.

The optimization problem consists of two steps:

1. Find the optimal terminal wealth
2. Find the corresponding trading strategy

Outline

Time-Dependent Convex Constraints

- We can write our model under full information with respect to $\mathcal{F}^R$ as

$$\mathrm{d}R_t = \hat{\mu}_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}V_t \, , \quad t \in [0, T] \, .$$

where the innovation process $V = (V_t)_{t \in [0,T]}$ is a $\mathbb{P}$-Brownian motion defined by

$$V_t = W_t + \int_0^t \sigma_s^{-1}(\mu_s - \hat{\mu}_s) \, \mathrm{d}s = \int_0^t \sigma_s^{-1} \, \mathrm{d}R_s - \int_0^t \sigma_s^{-1} \hat{\mu}_s \, \mathrm{d}s \, .$$

- K_t represents the constraints on portfolio proportions at time $t \rightarrow \eta_t^\pi \in K_t$
  K_t is a $\mathcal{F}_t$-progressively measurable closed convex set $\emptyset \neq K_t \subseteq \mathbb{R}^n$ that contains 0
- For each t we define the support function $\delta_t \colon \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ of $-K_t$ by

$$\delta_t(y) = \sup_{x \in K_t}(-x^\top y) \, , \quad y \in \mathbb{R}^n \, .$$

$\rightarrow$ $\delta_t(y)$ is $\mathcal{F}_t$-progressively measurable

$\rightarrow$ $y \mapsto \delta_t(y)$ is a lower semicontinuous, proper, convex function on its effective domain $\tilde{K}_t = \{y \in \mathbb{R}^n \colon \delta_t(y) < \infty\}$

Time-Dependent Convex Constraints

Definition

A trading strategy η^π is called K_t-admissible for initial capital $x_0 > 0$ if $X_t^\pi \geq 0$ a.s. and $\eta_t^\pi \in K_t$ for all $t \in [0, T]$.
We denote the class of admissible trading strategies for initial capital x_0 by $\mathcal{A}_{K_t}(x_0)$.

We introduce the set $\mathcal{H}$ of dual processes $\nu_t\colon [0, T] \times \Omega \mapsto \tilde{K}_t$ which are $\mathcal{F}_t^R$-progressively measurable processes, satisfying $\mathrm{E}\left[\int_0^T \left(\|\nu_t\|^2 + \delta_t(\nu_t)\right) \mathrm{d}t\right] < \infty$.
For each dual process $\nu \in \mathcal{H}$ we introduce

- a new interest rate process $r_t^\nu = r_t + \delta_t(\nu_t)$.
- a new drift process $\hat{\mu}_t^\nu = \hat{\mu}_t + \nu_t + \delta_t(\nu_t)\mathbf{1}_n$.
- a new market price of risk $\theta_t^\nu = \sigma_t^{-1}(\hat{\mu}_t - r_t + \nu_t)$
- a new density process ζ^ν given by $\mathrm{d}\zeta_t^\nu = -\theta_t^\nu \zeta_t^\nu \,\mathrm{d}V_t$

Then:

Solution under constraints **=** solution under no constraints with new market coefficients!

Problem:

Find optimal ν!

Time-Dependent Convex Constraints

Proposition

Suppose $x_0 > 0$ *and* $\mathrm{E}[U^-(X_T^\eta)] < \infty$ *for all* $\eta^\pi \in \mathcal{A}_K(x_0)$.

- *A trading strategy* $\eta^\pi \in \mathcal{A}_K(x_0)$ *is optimal, if for some* $y^* > 0$, $\nu^* \in \mathcal{H}$

  $$X_T^\pi = I(y^* \tilde{\zeta}_T^*) \, , \quad \mathcal{X}^{\nu^*}(y^*) = x_0 \, ,$$

  where $\tilde{\zeta}_T^* = \tilde{\zeta}_T^{\nu^*}$. *Further,* η^π *and* ν^* *have to satisfy the complementary slackness condition*

  $$\delta_t(\nu_t^*) + (\eta_t^\pi)^\top \nu_t^* = 0 \, , \quad t \in [0, T] \, .$$

- y^*, ν^* *solve the dual problem*

  $$\tilde{V}(y) = \inf_{\nu \in \mathcal{H}} \mathrm{E}\left[\tilde{U}(y \tilde{\zeta}_T^\nu)\right] \, ,$$

  where $\tilde{U}(y) = \sup_{x>0}\{U(x) - xy\}, y > 0$ *is the convex dual function of* U.

- *If* $\mathcal{F}^R = \mathcal{F}^V$ *holds, then an optimal trading strategy exists.*

Outline

Suppose we cannot trade in $[t, t+\Delta t]$. Then

$$\Delta X_t^\pi = X_{t+\Delta t}^\pi - X_t^\pi = X_t^\pi \exp\Big(\int_t^{t+\Delta t} r_s\,\mathrm{d}s\Big) - X_t^\pi + \exp\Big(\int_t^{t+\Delta t} r_s\,\mathrm{d}s\Big)(\eta_t^\pi)^\top X_t^\pi \\ \times \Big(\exp\Big(-\frac{1}{2}\int_t^{t+\Delta t} \mathrm{diag}(\sigma_s\sigma_s^\top)\,\mathrm{d}s + \int_t^{t+\Delta t} \sigma_s\,\mathrm{d}\tilde{W}_s\Big) - 1\Big) .$$

Next, we impose the relative LEL constraint

$$\tilde{\mathrm{E}}\big[(\Delta X_t^\pi)^- | \mathcal{F}_t^S\big] < \varepsilon_t ,$$

with $\varepsilon_t = L X_t^\pi$.

Definition

$$K_t^{LEL} := \big\{\eta_t^\pi \in \mathbb{R}^n \big| \tilde{\mathrm{E}}\big[(\Delta X_t^\pi)^- | \mathcal{F}_t^S\big] < \varepsilon_t\big\}$$

Limited Expected Loss & Limited Expected Shortfall

We introduce the relative LES constraint as an extension to the LEL constraint

$$\tilde{\mathrm{E}}\left[(\Delta X_t^\pi + q_t)^- | \mathcal{F}_t^S\right] < \varepsilon_t \ ,$$

with $\varepsilon_t = L_1 X_t^\pi$ and $q_t = L_2 X_t^\pi$.

- LES with $L_2 = 0$ corresponds to LEL with $L = L_1$.
- LEL: any loss in $[t, t + \Delta t]$ can be hedged with L% of the portfolio value.
- LES: any loss greater L_2% of the portfolio value in $[t, t + \Delta t]$ can be hedged with L_1% of the portfolio value.
- LEL & LES: For hedging we can use standard European call and put options.

Definition

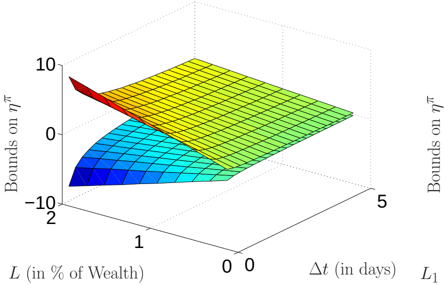
$$K_t^{LES} := \left\{\eta_t^\pi \in \mathbb{R}^n \middle| \tilde{\mathrm{E}}\left[(\Delta X_t^\pi + q_t)^- | \mathcal{F}_t^S\right] < \varepsilon_t\right\}$$

Lemma

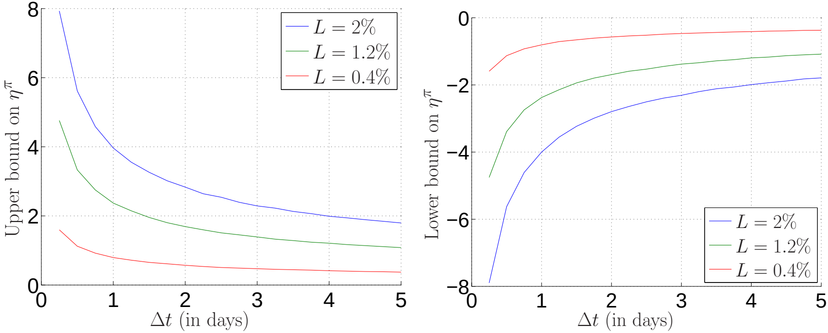
K_t^{LEL} and K_t^{LES} are convex.

For $n = 1$ we obtain the interval $K_t^{LES} = [\eta_t^l, \eta_t^u]$.

bounds on η^π for LEL and LES



bounds on η^π for LEL



Other constraints

Value-at-Risk constraint:

Under the original measure ΔX_t^π is given by

$$\Delta X_t^\pi = X_t^\pi \exp\Big(\int_t^{t+\Delta t} r_s \, \mathrm{d}s\Big) - X_t^\pi + (\eta_t^\pi)^\top X_t^\pi$$
$$\times \left(\exp\Big(\int_t^{t+\Delta t} (\mu_s - \frac{1}{2}\operatorname{diag}(\sigma_s\sigma_s^\top))\, \mathrm{d}s + \int_t^{t+\Delta t} \sigma_s \, \mathrm{d}W_s\Big) - \exp\Big(\int_t^{t+\Delta t} r_s \, \mathrm{d}s\Big)\right) .$$

We impose for $n = 1$ the relative VaR constraint on the loss $(\Delta X_t^\pi)^-$,

$$\mathrm{P}\big((\Delta X_t^\pi)^- > L X_t^\pi | \mathcal{F}_t^S, \mu_t = \hat{\mu}_t\big) < \gamma \, .$$

- VaR is computed under the original measure P.
- Under partial information we need the (unknown) value of the drift $\rightarrow$ use e.g. $\mu_t = \hat{\mu}_t$.
- For $n = 1$ we obtain the interval $K^{VaR} = [\eta_t^l, \eta_t^u]$.
- If $n > 2$ then K^{VaR} may not be convex!
- Possible to apply a large class of other risk constraints e.g. CVaR.

Corollary (Logarithmic utility)

$U(x) = \log(x)$, $n = 1$, *no constraints:*

$$\eta_t^o := \eta_t^\pi = \frac{1}{\sigma_t^2}(\hat{\mu}_t - r_t) \ .$$

With constraints:

$$\eta_t^c := \eta_t^\pi = \begin{cases} \eta_t^u & \text{if } \eta_t^o > \eta_t^u \ , \\ \eta_t^o & \text{if } \eta_t^o \in \left[\eta_t^l, \eta_t^u\right] \ , \\ \eta_t^l & \text{if } \eta_t^o < \eta_t^l \ . \end{cases}$$

Hence, we cut off the strategy obtained under no constraints if it exceeds or falls below a certain threshold.

Outline

Gaussian Dynamics (GD) for the Drift

- Drift: modeled as the solution of the stochastic differential equation (cf. Lakner '98)

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)\, dt + \upsilon\, d\bar{W}_t\,,$$

  $\mu_0 \sim \mathcal{N}(\hat{\mu}_0, \rho_0)$, n-dimensional,
  $\bar{W}$ is a n-dimensional Brownian motion with respect to $(\mathcal{F}, \mathrm{P})$,
- We are in the situation of Kalman-filtering with signal μ, observation R, and filter $\hat{\mu}_t = \mathrm{E}\big[\mu_t \,\big|\, \mathcal{F}_t^S\big]$.
- Filter: $\hat{\mu}_t$ is the unique $\mathcal{F}^S$-measurable solution of

$$d\hat{\mu}_t = \big[\big(-\kappa - \rho_t(\sigma_t\sigma_t^\top)^{-1}\big)\hat{\mu}_t + \kappa\bar{\mu}\big]\, dt + \rho_t(\sigma_t\sigma_t^\top)^{-1}\, dR_t\,,$$
$$\dot{\rho}_t = -\rho_t(\sigma_t\sigma_t^\top)^{-1}\rho_t - \kappa\rho_t - \rho_t\kappa^\top + \upsilon\upsilon^\top\,,$$

  with initial condition $(\hat{\mu}_0, \rho_0)$.
- ζ^{-1} satisfies $d\zeta_t^{-1} = \zeta_t^{-1}(\hat{\mu}_t - r_t\mathbf{1}_n)^\top(\sigma_t^\top)^{-1}\, d\tilde{W}_t$.

Proposition

$\mathcal{F}^S = \mathcal{F}^R = \mathcal{F}^{\tilde{W}} = \mathcal{F}^V \rightarrow$ *an optimal trading strategy exists.*

Bayesian case

The Bayesian case is a special case of the Gaussian dynamics for the drift.

- Drift: $\mu_t \equiv \mu_0 = (\mu_0^{(1)}, \dots, \mu_0^{(n)})$ is an (unobservable) $\mathcal{F}_0$-measurable Gaussian random variable with known mean vector $\hat{\mu}_0$ and covariance matrix ρ_0.
- Filter: Explicit solution:

$$\hat{\mu}_t = \left(\mathbf{1}_{n\times n} + \rho_0 \int_0^t (\sigma_s \sigma_s^\top)^{-1}\, \mathrm{d}s\right)^{-1} \left(\hat{\mu}_0 + \rho_0 \int_0^t (\sigma_s \sigma_s^\top)^{-1}\, \mathrm{d}R_s\right),$$

$$\rho_t = \left(\mathbf{1}_{n\times n} + \rho_0 \int_0^t (\sigma_s \sigma_s^\top)^{-1}\, \mathrm{d}s\right)^{-1} \rho_0 \,.$$

Outline

HMM: The Drift

The drift process μ of the return, is a continuous time Markov chain given by

$$\mu_t = BY_t \ , \qquad B \in \mathbb{R}^{n \times d} \ ,$$

where Y is a continuous time Markov chain with

- state space the standard unit vectors $\{e_1, \dots, e_d\}$ in $\mathbb{R}^d$, and
- rate matrix $Q \in \mathbb{R}^{d \times d}$, where
  - Q_{kl} is the jump rate or transition rate from e_k to e_l,
  - $\lambda_k = -Q_{kk} = \sum_{l=1, l \neq k}^{d} Q_{kl}$ is the rate of leaving e_k,
  - the waiting time for the next jump is exponentially distributed with parameter λ_k and Q_{kl}/λ_k is the probability that the chain jumps to e_l when leaving e_k for $l \neq k$.

The different states of the drift are the columns of B.

We can write the market price of risk as

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_n) = \Theta_t^\top Y_t \ , \quad \text{where} \quad \Theta_t := \sigma_t^{-1}(B - r_t \mathbf{1}_{n \times d}) \ .$$

We are in the situation of HMM filtering since $R_t = \int_0^t BY_s \, ds + \int_0^t \sigma_s \, dW_s$.

We need

- the conditional density $\zeta = (\zeta_t)_{t\in[0,T]} = \mathrm{E}\left[Z_t | \mathcal{F}_t^S\right] = \frac{1}{\mathbf{1}_d^\top \mathcal{E}_t}$,
- the unnormalized filter $\mathcal{E} = (\mathcal{E}_t)_{t\in[0,T]} = \tilde{\mathrm{E}}\left[Z_T^{-1} Y_t | \mathcal{F}_t^S\right]$,
- the normalized filter $\hat{Y} = (\hat{Y}_t)_{t\in[0,T]} = \mathrm{E}\left[Y_t | \mathcal{F}_t^S\right] = \frac{\mathcal{E}_t}{\mathbf{1}_d^\top \mathcal{E}_t} = \zeta_t \mathcal{E}_t$.

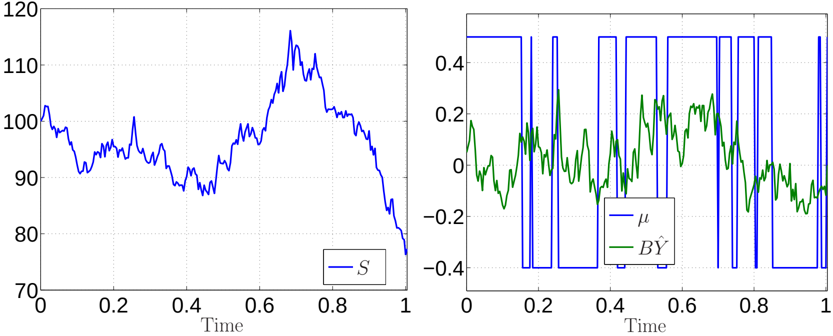
Theorem (Wonham/Elliott)

$$\mathcal{E}_t = \mathrm{E}[Y_0] + \int_0^t Q^\top \mathcal{E}_s \, ds + \int_0^t \mathrm{Diag}(\mathcal{E}_s) \Theta_s^\top \, d\tilde{W}_s$$

Proposition

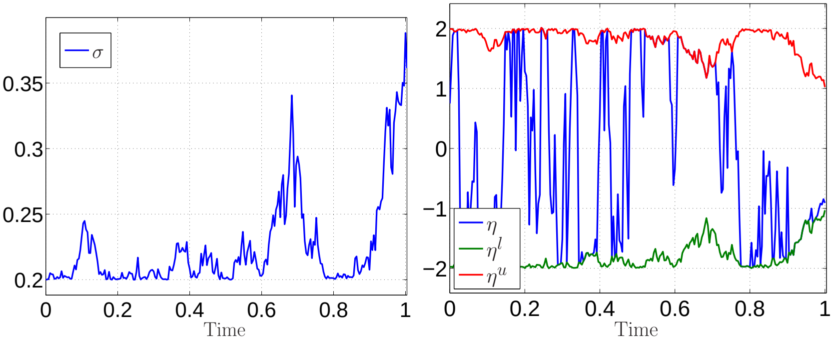
$\mathcal{F}^S = \mathcal{F}^R = \mathcal{F}^{\tilde{W}} = \mathcal{F}^V \rightarrow$ *an optimal trading strategy exists.*

Outline



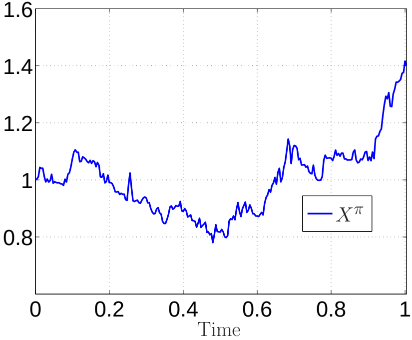
We consider the HMM for the drift.

Example ct'd (2/3)



For the volatility we consider the Hobson-Rogers model.

Example ct'd (3/3)



Numerical Results (1/2)

- We consider 20 stocks of the Dow Jones Industrial Index
- We use daily prices (adjusted for dividends and splits) for 30 years, 1972–2001
- Parameter estimates are based on five years with starting year 1972, 1973,..., 1996 using a Markov Chain Monte Carlo algorithm.
- We apply the strategy in the subsequent year
  $\rightarrow$ we perform 500 experiments whose outcomes we average.
- We consider LEL- and LES-constraint.

Numerical Results ct'd (2/2)

$U(\hat{X}_T)$	mean	median	st.dev.	aborted
unconstrained				
b&h	**0.1188**	0.1195	0.2297	0
Merton	0.0248	0.0826	0.4815	2
GD	-1.2002	-1.0000	0.9580	79
Bayes	0.0143	0.0824	0.5071	2
HMM	-0.0346	0.0277	0.9247	13
LEL risk constraint (L=0.5%)				
GD	0.0252	0.0294	0.1767	0
Bayes	0.1002	0.0988	0.1595	0
HMM	**0.1285**	0.1242	0.2004	0
LES risk constraint (L1=0.1%,L2=5%)				
GD	-0.0395	-0.0350	0.3086	0
Bayes	0.0950	0.0968	0.2752	0
HMM	**0.1505**	0.1402	0.3434	0

- LEL and LES improve the performance of all models.
- With LEL and LES we don't go bankrupt anymore.
- The HMM strategy with risk constraints outperforms all other strategies.

Conclusion & Outlook

Conclusion

- *We show how to apply dynamic risk constraints using time-dependent convex constraints.*
- *We derive explicit trading strategies with dynamic risk constraints under partial information.*
- *The numerical results indicate that dynamic risk constraints can reduce the risk and improve the performance.*

Outlook

- *Allow for consumption.*
- *More detailed analysis of the multidimensional case.*
- *Explicit strategies for general utility.*

Further Reading

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