Inflation-indexed Swaps and Swaptions

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Outline

- Introduction: Markets, Instruments & Literature
- Foreign-Exchange Analogy
- Pricing & Hedging of Inflation Swaps
- Pricing Inflation Swaptions with a Market Model
Overview

1. Introduction to the Inflation Market & Instruments
2. Foreign-Exchange Analogy
3. Pricing & Hedging of Inflation Swaps
4. Pricing Inflation Swaptions with a Market Model
Introduction to the Inflation Market & Instruments

Inflation

- An increase in the economy’s price level is known as inflation. Inflation reduces the purchasing power, i.e. the value of money decrease.

- A consumer price index (CPI) is the price of a particular basket consisting of consumer goods and services. The price index is a measure of the general price level in the economy.

- Inflation is typically measured as the percentage rate at which the consumer price index changes over a certain period of time.

- Negative inflation is known as deflation.
Introduction to the Inflation Market & Instruments

Inflation Protected Bonds

- Many names: Inflation-indexed bonds, Inflation linked bonds, Real bonds, TIPS (US), Index-linked gilts (UK).
- The payoff is linked to a price index. (CPI, RPI)
- Typically coupon bonds.
- Can be floored.
- The issuer of an inflation protected bond has an incentive to keep inflation low. Useful for governments.
- These bonds are typically issued by Treasuries.
- Typical investors are pension funds, mutual funds.
- World wide outstanding nominal amount 2007: 1000 billion dollar.
Markets

- UK (1981)
- Australia (1983)
- Canada (1991)
- Sweden (1994)
- United States (1997)
- Greece (1997)
- France (1998)
- Italy (2003)
- Japan (1904)
- Germany (2006)

Earlier: Chile, Brazil, Columbia, Argentina.

First inflation protected bond issue: Massachusetts Bay Company 1780.
Inflation Derivatives

- Swaps
- Caps & Floors
- Swaptions
- Bond options
- ...
Introduction to the Inflation Market & Instruments

Inflation Indexed Swaps & Swaptions

Inflation Indexed Swap

- Agreement between two parties A and B to exchange cash flows in the future
- Prespecified dates for when the cash flows are to be exchanged
- At least one of the cash flows is linked to inflation (CPI)

\[ \text{Diagram: } A \rightarrow B \]

Inflation Indexed Swaption

- It is an option to enter into an inflation indexed swap at pre specified date at a pre determined swap rate.
Main References

Hughston (1998)
- General theory
- Foreign-currency analogy

Jarrow & Yildirim (2003)
- 3-factor HJM model
- TIPS (coupon bonds)
- Option on Inflation index

Mercurio (2005)
- YYIIS, Caplets, Floorlets (ZCIIS)
- JY version of HJM with Hull-White vol
- 2 Market Models
Contribution

- HJM model with jumps
  - YYIIS

Inflation Swap Market Models
- ZCIISwaptions
- YYIISwaptions

- HJM model
  - ZCIISwaptions
  - TIPStions

- Verify the foreign-currency analogy for an arbitrary process

YYIIS = Year-on-Year Inflation Indexed Swaps
ZCIIS = Zero Coupon Inflation Indexed Swaps
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Price and Payoff

$I(t) : \quad$ An arbitrary stochastic process

$p^n(t, T) : \quad$ Price in dollar at $t$ of a contract that pays out 1 dollar at $T$.

$p^{IP}(t, T) : \quad$ Price in dollar at $t$ of a contract that pays out $I(T)$ dollar at $T$.

Assume : There exist a market for $p^n(t, T)$ and $p^{IP}(t, T)$ for all $T$

Define : $p^r(t, T) = \frac{p^{IP}(t, T)}{I(t)}$
If \( I(t) \) is the price of a hamburger

A nominal bond:
- Pays out 1 dollar at maturity.
- \( p^n(t, T) \): the price of a nominal bond is in dollar

A hamburger-indexed bond:
- At maturity it pays out a dollar amount that is enough to buy 1 hamburger.
- \( p^{IP}(t, T) \): the price of a hamburger-inflation protected bond is in dollar

A hamburger-real bond:
- Pays out 1 hamburger at maturity
- \( p^r(t, T) \): the price of a real bond is in hamburgers

Note: CPI
Define

**Forward rates:**

\[ f^i(t, T) = -\frac{\partial \ln p^i(t, T)}{\partial T} \quad \text{for } i = r, n. \]

**Short rates:**

\[ r^i(t) = f^i(t, t) \quad \text{for } i = r, n. \]

**Money Market Accounts:**

\[ B^i(t) = e^{\int_0^t r^i(s)ds} \quad \text{for } i = r, n. \]
HJM model with Jumps

Assume:
Under the objective probability measure $P$:

\[
\begin{align*}
    df_t^r(T) &= \alpha_t^r(T)dt + \sigma_t^r(T)dW^P + \int_V \xi^r(t,v,T)\mu^P(dt,dv) \\
    df_t^n(T) &= \alpha_t^n(T)dt + \sigma_t^n(T)dW^P + \int_V \xi^n(t,v,T)\mu^P(dt,dv) \\
    dI_t &= I_t\mu_t^I dt + I_t\sigma_t^I dW^P + I_t- \int_V \gamma_t^I(v)\mu^P(dt,dv)
\end{align*}
\]
Now calculate

1. Forward rates ⇒ Bondprices (BKR)

2. Change measure from $P$ to $Q^n$ (Girsanov)
   Now we have found the $Q^n$-drift of $p^n(t, T)$ and $p^{IP}(t, T)$ which we call $\mu^n_Q(t, T)$ and $\mu^{IP}_Q(t, T)$

3. By requiring

   \[ \frac{p^n(t, T)}{B^n(t)} \quad \frac{p^{IP}(t, T)}{B^n(t)} \]
   are $Q^n$-martingales

i.e. $\mu^n_Q(t, T) = \mu^{IP}_Q(t, T) = r^n(t)$ for all maturities $T$.  

3 drift conditions

One of the 3 conditions tells us that the $Q^n$-drift of the index $I$ is equal to $r^n - r^r$.
Three drift conditions

\[
\alpha^n(t, T) = \sigma^n(t, T) \left( \int_t^T \sigma^r(t, s) ds - h(t) \right) \\
- \int_V \{ \delta^n(t, v, T) + 1 \} \xi^n(t, v, T) \lambda_t(dv)
\]

\[
\alpha^r(t, T) = \sigma^r(t, T) \left( \int_t^T \sigma^r(t, s) ds - \sigma^I(t) - h(t) \right) \\
- \int_V \left( 1 + \gamma^I(t, v) \right) (1 + \delta^r(t, v, T)) \xi^r(t, v, T) \lambda_t(dv)
\]

\[
\mu^I(t) = r^n(t) - r^r(t) - h(t)\sigma^I(t) - \int_V \gamma^I(t, v) \lambda_t(dv)
\]
Result

Under the nominal risk neutral measure $Q^n$:

\[
\frac{dp_t^n(T)}{p_{t-}^n(T)} = r_t^n dt + \beta_t^n(T) dW + \int_V \delta_t^n(v, T) \tilde{\mu}(dt, dv)
\]

\[
\frac{dp_t^{IP}(T)}{p_{t-}^{IP}(T)} = r_t^{IP} dt + \beta_t^{IP}(T) dW + \int_V \delta_t^{IP}(v, T) \tilde{\mu}(dt, dv)
\]

\[
\frac{dI_t}{I_{t-}} = (r_t^n - r_t^r) dt + \sigma_t^I dW + \int_V \gamma_t^I(v) \tilde{\mu}(dt, dv)
\]

\[
\frac{dp_t^r(T)}{p_{t-}^r(T)} = a(t, T) dt + \beta_t^r(T) dW + \int_V \delta_t^r(v, T) \tilde{\mu}(dt, dv)
\]

where

\[
\tilde{\mu}(dt, dv) = \mu(dt, dv) - \lambda_t(dv) dt
\]

Note: $I$ has the same dynamics as an FX-rate!
Foreign Currency Analogy

Nominal vs Real

\[ p^n(t, T) : \text{Price of nominal } T\text{-bond in dollar} \]
\[ p^r(t, T) : \text{Price of real } T\text{-bond in CPI units}^* \]
\[ I(t) : \text{Price level (dollar per CPI-unit)} \]
\[ p^{IP}(t, T) : \text{Price of a real } T\text{-bond in dollar denoted by } p^{IP}(t, T) \]

Domestic vs Foreign

\[ p^n(t, T) : \text{Price of domestic } T\text{-bond} \]
\[ p^r(t, T) : \text{Price of foreign } T\text{-bond} \]
\[ I(t) : \text{FX-rate (domestic per foreign unit)} \]
\[ I(t)p^r(t, T) : \text{Domestic price of foreign } T\text{-bond}^* \]
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Assumptions

- Exist a market for nominal T-bonds and nominal indexed-bonds for all maturity dates.
- The bond prices are differentiable wrt $T$.
- Forward rate dynamics according to HJM with jumps.
- Existence of martingale measure.
- All volatilities and the intensity are deterministic under the nominal risk neutral measure.
A Payer Swap

- starts at time $T_m$

- At each payment date $T_j$ where $j = m + 1, m + 2, \cdots, T_M$
  
  ▶ you pay

  $\alpha_j K$

  ▶ you receive

  $\alpha_j \left[ \frac{I(T_j)}{I(T_{j-1})} - 1 \right]$
The price

The price at time \( t \) is:

\[
\sum_{j=m+1}^{M} \Pi \left[ t, \alpha_j \frac{I(T_j)}{I(T_{j-1})} \right] - (K + 1) \sum_{j=m+1}^{M} \alpha_j p(t, T_j)
\]

Find:

\[
\Pi \left[ t, \alpha_j \frac{I(T_j)}{I(T_{j-1})} \right]
\]

i.e. the price of payoff

\[
\alpha_j \frac{I(T_j)}{I(T_{j-1})}
\]
Let $M(t)$ and $N(t)$ be two martingales so that

\[ E_t[M_T] = M_t \]
\[ E_t[N_T] = N_t \]

The key to price this swap is to find:

\[ E_t[M_T N_T] \]

Solution:

- If $M(t)$ and $N(t)$ are independent, then $E_t[M_T N_T] = M_t N_t$
- What if they are NOT independent?
  \[ E_t[M_T N_T] = M_t N_t G^T_t \text{ where } G^T_t \text{ is the "convexity correction".} \]
Basic Result

Let $M(t)$ and $N(t)$ be two martingales with dynamics:

\[
\frac{dM_t}{M_t} = \sigma^M_t \, dW_t + \int_V \delta^M_t(v) \tilde{\mu}(dt, dv)
\]

\[
\frac{dN_t}{N_t} = \sigma^N_t \, dW_t + \int_V \delta^N_t(v) \tilde{\mu}(dt, dv)
\]

Assume that $\sigma^M, \sigma^N, \delta^M, \delta^N, \lambda$ are deterministic.

Then

\[
E_t[M_T N_T] = M_t N_t G^T_t
\]

where

\[
G^T_t = e^{\int_t^T (\sigma^M_u \cdot \sigma^N_u + \int_V \delta^M_u(v) \delta^N_u(v) \lambda_u(dv)) \, du}
\]
The Inflation-linked Swap Leg

The payoff function

\[ x_2 = \frac{I(T_2)}{I(T_1)} \]

The value at time \( t \)

\[ \Pi [t, x_2] = p^n(t, T_1) E^{T_1,n}_t [p^r(T_1, T_2)] \]

where

\[ E^{T_1,n}_t [p^r(T_1, T_2)] = E^{T_1,r}_t \left[ \frac{p^r(T_1, T_2)}{p^r(T_1, T_1)} \frac{L(T_1)}{L(t)} \right] = \frac{p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

hence

\[ \Pi [t, x_2] = \frac{p^n(t, T_1)p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

where \( C(t, T_1, T_2) \) is a convexity correction term.
The Inflation-linked Swap Leg

The payoff function

\[ \chi_2 = \frac{I(T_2)}{I(T_1)} \]

The value at time \( t \)

\[ \Pi [t, \chi_2] = p^n(t, T_1) E^{T_1,n}_t [p^r(T_1, T_2)] \]

where

\[ E^{T_1,n}_t [p^r(T_1, T_2)] = E^{T_1,r}_t \left[ \frac{p^r(T_1, T_2)}{p^r(T_1, T_1)} \frac{L(T_1)}{L(t)} \right] = \frac{p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

hence

\[ \Pi [t, \chi_2] = \frac{p^n(t, T_1) p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

where \( C(t, T_1, T_2) \) is a convexity correction term.
The Inflation-linked Swap Leg

The payoff function

\[ x_2 = \frac{I(T_2)}{I(T_1)} \]

The value at time t

\[ \Pi [t, \chi_2] = p^n(t, T_1) E^{T_1,n}_t [p^r(T_1, T_2)] \]

where

\[ E^{T_1,n}_t [p^r(T_1, T_2)] = E^{T_1,r}_t \left[ \frac{p^r(T_1, T_2)}{p^r(T_1, T_1)} \frac{L(T_1)}{L(t)} \right] = \frac{p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

hence

\[ \Pi [t, \chi_2] = \frac{p^n(t, T_1) p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2) \]

where \( C(t, T_1, T_2) \) is a convexity correction term.
The Payer Swap

The price of the swap is at time $t$

$$\sum_{j=m+1}^{M} \alpha_j p^n(t, T_{j-1}) p^{IP}(t, T_j) C(t, T_{j-1}, T_j) \frac{p^{IP}(t, T_{j-1})}{p^{IP}(t, T_{j-1})}$$

$$- (K + 1) \sum_{j=m+1}^{M} \alpha_j p^n(t, T_j)$$

Note

The price does only depend on bonds in the nominal market.
Hedging the Inflation-linked Swap leg I

When no jumps

\[ \Pi \left[ t, X_2 \right] = \frac{p^n(t, T_1)p^{IP}(t, T_2)}{p^{IP}(t, T_1)} e^{g(s,T_1,T_2)} \]

where

\[ g(s, T_1, T_2) = \int_t^{T_1} (\beta^n(s, T_1) - \beta^{IP}(s, T_1)) \cdot (\beta^{IP}(s, T_2) - \beta^{IP}(s, T_1)) \, ds \]

Idea:
Try to replicate the swap leg using \( p^n(t, T_1), p^{IP}(t, T_2) \) and \( p^{IP}(t, T_1) \) from now on referred to as \( S_1, S_2 \) and \( S_3 \) respectively.
Hedging the Inflation-linked Swap leg II

Define portfolio strategy \((h_1(t), h_2(t), h_3(t))\) for \(t \leq T_1\) as

\[
h_i(t) = \frac{\Pi [t, \mathcal{X}_2]}{S_i(t)} \quad \text{for } i = 1, 2, \quad h_3(t) = -\frac{\Pi [t, \mathcal{X}_2]}{S_3(t)}
\]

Then for \(t \leq T_1\)

\[
V^h(t) = \sum_{i=1}^{3} h_i S_i = \Pi(t)
\]

\[
dV^h(t) = \sum_{i=1}^{3} h_idS_i
\]

At time \(T_1\) we have \(V^h(T_1) = \frac{p^{IP}(T_1, T_2)}{I(T_1)}\) which is just enough to buy \(\frac{1}{I(T_1)}\) \(T_2 - IP\)-bonds which we keep until maturity and thus results in \(\frac{I(T_2)}{I(T_1)}\) as it should!
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The Swap rate

Recall that the swap price is:

\[ YYIIS^M_{m}(t, K) = \sum_{j=m+1}^{M} \Pi \left[ t, \alpha_j \frac{I(T_j)}{I(T_{j-1})} \right] - (K + 1)S^M_{m}(t) \]

where

\[ S^k_{m}(t) = \sum_{j=m+1}^{k} \alpha_j \rho_n(t, T_j) \]

The par swap rate is:

\[ R^M_{m}(t) = \sum_{j=m+1}^{M} \frac{\Pi \left[ t, \alpha_j \frac{I(T_j)}{I(T_{j-1})} \right]}{S^M_{m}(t)} - S^M_{m}(t) \]
The HJM Swap rate

The par swap rate at time t

\[ R^M_m(t) = \frac{\sum_{j=m+1}^{M} \alpha_j p^n(t,T_{j-1}) p_{IP}(t,T_j) C(t,T_{j-1},T_j)}{S^M_m(t)} - S^M_m(t) \]

Where

\[ S^k_m(t) = \sum_{j=m+1}^{k} \alpha_j p_n(t,T_j) \]

Note

Nasty distribution!
A payer YYIIISwaption

Payoff function

\[ \Upsilon^M_m = \max[YYIIIS^M_m(T_m, K), 0] \]

Rewritten Payoff function

\[ \Upsilon = S_T \max[R_T - K, 0] \]

where

- \( R_T \) par swap rate (self-financing portfolio)
- \( S_T \) sum of nominal bonds (self-financing portfolio)

Note

Easy if \( R_T \) is lognormal! \( \Rightarrow \) Black’s pricing formula
**Swap Market Model**

**Definition:**
For any given pair \((m, k)\) of integers s.t. \(0 \leq m < k < M\) we assume that, under the measure for which \(S^k_m\) is numeraire, the forward swap rate \(R^k_m\) has dynamics given by

\[
dR^k_m(t) = R^k_m(t)\sigma^k_m(t)dW^k_m(t)
\]

where \(\sigma^k_m(t)\) is deterministic
Summarizing

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Remarks

- TIPS can have embedded option features
- Reverse: Swaps as given price TIPS
- The CPI index used is typically lagged
- The CPI index is typically only observed monthly (linearly interpolated)
- The suggested market model is not proved to exist