Early exercise boundary regularity close to expiry in indifference setting

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A contract on one or several underlying assets that can be exercised during some predetermined period \([t, T]\).
American option

- A contract on one or several underlying assets that can be exercised during some predetermined period $[t, T]$.
- Payoff $g : \mathbb{R}^n \to \mathbb{R}$ at exercise $\tau \in [t, T]$. 
Example: American put option

Gives you the right, but not the obligation, to sell the underlying stock $X_s$ for a predetermined price $K$ any time $s \in [t, T]$. 
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At exercise $\tau$ the payoff is $g(X_\tau) = \max(K - X_\tau, 0)$. 

![Graph showing the payoff of an American put option](image)
Complete markets

The market consists of

- non-risky asset

\[ dB_s = \rho B_s ds \]
\[ B_t = B. \]

- traded asset

\[ dX_s = \mu X_s ds + \sigma X_s dW_s \]
\[ X_t = x \]

\( W_s \) is Brownian motion.
Option price

The price $h$ of an American option with payoff $g$ is given by

**Theorem (Risk-neutral valuation formula)**

\[ h(x, t) = \sup_{\tau \in [t, T]} e^{-\rho(\tau-t)} E(g(X_\tau) | X_t = x). \]
Variational inequality

$h$ solves the following linear variational inequality

$$\min \left( -h_t - \frac{1}{2}\sigma^2 x^2 h_{xx} - \rho x h_x + \rho h, h(x, t) - g(x) \right) = 0 \quad \text{in} \quad \mathbb{R} \times [0, T)$$

$$h(x, T) = g(x) \quad \text{in} \quad [0, T)$$
Variational inequality

$h$ solves the following linear variational inequality

$$\min \left( -h_t - \frac{1}{2} \sigma^2 x^2 h_{xx} - \rho \cdot x h_x + \rho h, \quad h(x, t) - g(x) \right) = 0 \quad \text{in} \quad \mathbb{R} \times [0, T)$$

$$h(x, T) = g(x) \quad \text{in} \quad [0, T)$$

A free boundary $\Gamma$ separates the sets

$$\mathcal{C} = \{ -h_t - \frac{1}{2} \sigma^2 x^2 h_{xx} - \rho \cdot x h_x + \rho h = 0 \}$$

$$\mathcal{E} = \{ h - g = 0 \}.$$
History

Independent results for the American put.

- Kuske & Keller (1998)
- Bunch & Johnsson (2000)
- Stamicar, Sevcovic & Chadam (1999)
Chen, Chadam: Reformulation

In dimensionless variables the price function $\tilde{h}(x, t)$ solves

$$
\tilde{h}_t - \tilde{h}_{xx} - (k - 1)\tilde{h}_x + k\tilde{h} = 0 \quad \text{for} \quad x > \tilde{\beta}(t)
$$

$$
\tilde{h} = 1 - e^x \quad \text{for} \quad x < \tilde{\beta}(t)
$$

$$
\tilde{h}(0, x) = (1 - e^x)^+,
$$

where $x = \tilde{\beta}(t)$ is a parameterization of the free boundary $\Gamma$. 

Fundamental solution

Find the fundamental solution for the PDE

\[ \Phi(x, t) = \frac{1}{2\sqrt{\pi t}} \exp \left\{ -\frac{(x + (k - 1)t)^2}{4t} \right\} \]

and get the following integral representation

\[ \tilde{h}(x, t) = \int_{-\infty}^{0} (1 - e^y) \Phi(x - y, t) dy \]

\[ + k \int_{0}^{t} \int_{-\infty}^{\beta(t-\theta)} \Phi(x - y, \theta) dy d\theta. \]
ODE for the free boundary

Derive an ODE for the free boundary

\[
\dot{\tilde{\beta}} = -\frac{2\Phi_x(\tilde{\beta}(t), t)}{k} - 2 \int_0^t \Phi_x(\tilde{\beta}(t) - \tilde{\beta}(t - \theta), \theta) \dot{\tilde{\beta}}(t - \theta) d\theta.
\]

Asymptotic expansion

\[
\tilde{\beta} = -\xi - \frac{1}{2} \xi + \frac{1}{8} \xi^2 + \frac{17}{24} \xi^3 + \ldots
\]

where \(\xi = \sqrt{\frac{4}{k^2}}\).
ODE for the free boundary

Derive an ODE for the free boundary

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\dot{\beta} = -\frac{2\Phi_x(\tilde{\beta}(t), t)}{k} - 2 \int_0^t \Phi_x(\tilde{\beta}(t) - \tilde{\beta}(t - \theta), \theta) \dot{\beta}(t - \theta) d\theta.
\]

Asymptotic expansion

\[
\frac{\beta^2}{4t} = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{17}{24\xi^3} + \ldots
\]

where \( \xi = \sqrt{4\pi k^2 t} \).
Summary of the expansion method

Advantage

- Good precision

Drawback

- One-dimensional, linear setting.
A general obstacle problem

Obstacle problem with a non-linear, \( n + 1 \)-dimensional, parabolic operator

\[
\min(D_t u - F(D^2 u, Du, u, x, t), u - g) = 0 \quad \text{in } B_1 \times (0, 1)
\]

\[
u(x, 0) = g(x) \quad \text{in } B_1
\]

where \( B_1 \) is the unit ball in \( \mathbb{R}^n \).
Scaling in the point \((0,0)\)

For simplicity assume: \(u(0,0) = g(0) = 0\).

Scaled function

\[
    u_r(x, t) = \frac{u(rx, r^2t)}{\alpha_r}
\]

Scaled operator

\[
    F_r(D^2u, Du, u, x, t) = F(D^2u, rDu, r^2u, rx, r^2t).
\]
Scaling in the point \((0,0)\)

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u_r(x, t) = \frac{u(rx, r^2 t)}{\alpha_r}\]

Scaled operator

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F_r(D^2 u, Du, u, x, t) = F(D^2 u, rDu, r^2 u, rx, r^2 t).
\]

Choose \(\alpha_r\) so that \(0 < \lim_{r \to 0} u_r < \infty\) .
Scaled obstacle problem

Under standard assumptions on $F$ the scaled function $u_r$ solves

$$\min(D_t u_r - F_r(D^2 u_r, Du_r, u_r, x, t),$$

$$u_r - g_r = 0 \quad \text{in } B_{1/r} \times (0, \frac{1}{r^2})$$

$$u_r(x, 0) = g_r(x) \quad \text{in } B_{1/r}.$$
Blow-up limit

Take the so called *blow-up limit* by letting $r \to 0$.

If we have the right growth and continuity of $u$ the limit function $u_0 = \lim_{r \to 0} u_r$ will solve

$$\min(D_t u_0 - F(D^2 u_0, 0, 0, 0, 0), u_0 - g_0) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+$$

$$u_0(x, 0) = g_0(x) \quad \text{in } \mathbb{R}.$$
Free boundary regularity

Assume we have a free boundary.

\[ u = g \]

\[ D_t u - F u = 0 \]
Free boundary regularity

Assume that the free boundary stays above \( t = cx^2 \).
Free boundary regularity

Pick a sequence $X_1, X_2 \ldots \in \{t = cx^2\}$, where $X_j = (x_j, t_j)$. 

\[ u = g \quad D_t u - F u = 0 \]
Free boundary regularity

Set \( r_j = |X_j| \ldots \)

\[ D_t u - F u = 0 \]
Free boundary regularity

... and scale the problem by $r_j$. $\tilde{X}_j = (x_j/r_j, t_j/r_j^2)$. 

$$D_t u_{r_j} - F_{r_j} u_{r_j} = 0$$

$$B_1 \times (0, 1)$$

$$B_{r_j} \times (0, r_j^2)$$

$u_{r_j} = g_{r_j}$
Free boundary regularity

Take the limit as \( j \to \infty \). Note \( |\tilde{X}_\infty| = 1 \).
The blow-up limit problem

- For the limit problem no lower order terms occur in the PDE.
- The limit obstacle $g_0$ is possibly simpler than the original $g$. 

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Different scenarios that might occur for the limit problem:

- The obstacle is a strict subsolution to the differential operator.
- We can find an analytic solution.
The blow-up limit problem

- For the limit problem no lower order terms occur in the PDE.
- The limit obstacle $g_0$ is possibly simpler than the original $g$.

\[\Downarrow\]

Different scenarios that might occur for the limit problem:

- The obstacle is a *strict* subsolution to the differential operator.
- We can find an analytic solution.
The obstacle is a strict subsolution

$g_0$ is a strict subsolution if

$$-F(D^2g_0, 0, 0, 0, 0) < 0 \text{ in } B_1 \times (0, 1).$$
The obstacle is a strict subsolution

$g_0$ is a strict subsolution if

\[-F(D^2 g_0, 0, 0, 0, 0) < 0 \text{ in } B_1 \times (0, 1).\]

$D_t u_0 - F(u_0, 0, 0, 0, 0) \geq 0 \text{ in } B_1 \times (0, 1)$ and the maximum principle

\[\downarrow\]

$u_0 > g_0 \text{ in } B_1 \times (0, 1).$
The obstacle is a strict subsolution

g_0\text{ is a strict subsolution if }

\[ -F(D^2 g_0, 0, 0, 0, 0) < 0 \text{ in } B_1 \times (0, 1). \]

\[ D_t u_0 - F(u_0, 0, 0, 0, 0) \geq 0 \text{ in } B_1 \times (0, 1) \]

and the maximum principle

\[ \Downarrow \]

\[ u_0 > g_0 \text{ in } B_1 \times (0, 1). \]

\[ \Downarrow \]

No free boundary exists for the limit problem, i.e.

\[ \Gamma \in \{ t < x^2 \cdot \sigma(x) \} \]

for some modulus of continuity \( \sigma(x) \).
Incomplete markets: Market components

The market consists of

- non-risky asset (zero interest rate for simplicity)
  \[ B_s = B. \]

- traded asset
  \[ dX_s = \mu X_s ds + \sigma X_s dW_s \]
  \[ X_t = x \]

- non-traded asset
  \[ dY_s = b(Y_s, s) ds + a(Y_s, s) dW'_s \]
  \[ Y_t = y \]

\( W_s \) and \( W'_s \) are correlated with correlation \( \rho \in (-1, 1) \).
Aim

Define the *indifference price* $h$ of a call option written on the non-traded asset $Y_s$. 
Investment alternatives

**Alternative 1**: Invest in stock $X_s$ and bond $B_s$

- Allocation in traded stock $X_s$: $\pi_s$
- Allocation in bond: $\pi^0_s$
- Wealth: $Z_s = \pi^0_s + \pi_s$.

\[
dZ_s = \pi_s \mu ds + \pi_s \sigma dW_s
\]

\[
Z_t = z.
\]
Investment alternatives

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- Allocation in traded stock $X_s$: $\pi_s$
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- Wealth: $Z_s = \pi^0_s + \pi_s$.  

\[
dZ_s = \pi_s \mu ds + \pi_s \sigma dW_s
\]

\[
Z_t = z.
\]

**Alternative 2:** Invest in stock $X_s$, bond $B_s$ and buy a call option on non-traded asset $Y_s$ at time $t$ for price $h$

- American call payoff: $g(y) = (y - K)^+$.  

Indifference pricing

Alternative 1 (Stock and bond only)

- Initial wealth: \( z \)
- Terminal wealth: \( Z_T \)
- Value function:

\[ V_1(z, t) = \sup_{\pi} E(U(Z_T)|Z_t = z). \]

where \( U(z) = -e^{-\gamma z} \).
Indifference pricing

- **Alternative 1** (Stock and bond only)
  Initial wealth: $z$
  Terminal wealth: $Z_T$
  Value function:
  \[ V_1(z, t) = \sup_{\pi} E(U(Z_T)|Z_t = z). \]
  where $U(z) = -e^{-\gamma z}$.

- **Alternative 2** (Stock, bond and call option)
  Initial wealth: $z - h$
  Wealth at exercise time $\tau$: $Z_\tau + g(Y_\tau)$
  Value function:
  \[ V_2(z, y, t) = \sup_{\pi, \tau} E(V_1(Z_\tau + g(Y_\tau), \tau)|Z_\tau = z, Y_\tau = y) \]
Indifference pricing

- **Alternative 1** (Stock and bond only)
  - Initial wealth: \( z \)
  - Terminal wealth: \( Z_T \)
  - Value function:
    \[
    V_1(z, t) = \sup_{\pi} E(U(Z_T)|Z_t = z).
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    where \( U(z) = -e^{-\gamma z} \).

- **Alternative 2** (Stock, bond and call option)
  - Initial wealth: \( z - h \)
  - Wealth at exercise time \( \tau \): \( Z_\tau + g(Y_\tau) \)
  - Value function:
    \[
    V_2(z, y, t) = \sup_{\pi, \tau} E(V_1(Z_\tau + g(Y_\tau), \tau)|Z_\tau = z, Y_\tau = y)
    \]

- Definition: The indifference price \( h \) satisfies
  \[
  V_1(z, t) = V_2(z - h, y, t)
  \]
Early exercise boundary regularity close to expiry in indifference setting

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American options
History and background
The blow-up technique
Application to indifference pricing

Variational inequality

\[
\min(\mathcal{H} h, h - g) = 0 \quad \text{in } \mathbb{R} \times [0, T)
\]
\[
h(y, T) = g(y) \quad \text{in } \mathbb{R}
\]

where

\[
\mathcal{H} u = D_t u - \frac{1}{2} a^2(y, t) D_y^2 u - \left( b(y, t) - \rho \frac{\mu}{\sigma} a(y, t) \right) D_y u
\]
\[
+ \frac{1}{2} \gamma (1 - \rho^2) a^2(y, t) (D_y u)^2.
\]
Free boundary at expiry

- Parameterization of free boundary: $\Gamma = (\beta(t), t)$
- Location at expiry: $\beta_0 = \lim_{t \to 0} \beta(t)$
- $A(y, t) = -\mathcal{H}g^\text{call} = b - \rho \frac{\mu}{\sigma} a - \frac{1}{2} \gamma (1 - \rho^2) a^2$
Free boundary at expiry

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Lemma 1 If $A(y_0, 0) = 0$ and $A(y_0 + \delta, 0) A(y_0 - \delta, 0) < 0$ for all small $\delta$ then either no free boundary exists or $\beta_0 = y_0$. 

\[ A < 0 \quad A > 0 \quad y_0 \quad k \]
Free boundary at expiry

- Parameterization of free boundary: $\Gamma = (\beta(t), t)$
- Location at expiry: $\beta_0 = \lim_{t \to 0} \beta(t)$
- $A(y, t) = -\mathcal{H}g^{\text{call}} = b - \rho \mu \sigma a - \frac{1}{2} \gamma (1 - \rho^2) a^2$

**Lemma 1** If $A(y_0, 0) = 0$ and $A(y_0 + \delta, 0)A(y_0 - \delta, 0) < 0$ for all small $\delta$ then either no free boundary exists or

$$\beta_0 = y_0.$$

**Lemma 2** If $A(y, 0) < -\varepsilon$ for some $\varepsilon > 0$ and all $y \in \{g > 0\}$ then

$$\beta_0 = K.$$
Free boundary regularity: $\beta_0 \neq K$

**Theorem 1** There exists $\xi_0$ and $r > 0$ such that for $\xi_1 < \xi_0^{-2} < \xi_2$ and $t < r$

$$\{(\beta(t), t) : \xi_1(y - \beta_0)^2 \leq t \leq \xi_2(y - \beta_0)^2\}.$$
Free boundary regularity: $\beta_0 \neq K$

**Theorem 1** There exists $\xi_0$ and $r > 0$ such that for $\xi_1 < \xi_0^{-2} < \xi_2$ and $t < r$

$$(\beta(t), t) \in \{(y, t) : \xi_1(y - \beta_0)^2 \leq t \leq \xi_2(y - \beta_0)^2\}.$$ 

$\xi_0$ solve $u(\xi_0) - \xi_0 u'(\xi_0) = 0$ where

$$u(\xi) = \xi(6a^2(\beta_0, 0) + \xi^2) \int_{-\infty}^{\xi} \exp\left(\frac{-x^2}{4a^2(\beta_0, 0)}\right) \frac{1}{(6a^2(\beta_0, 0) + x^2)^2} \, dx.$$
Proof

★ Rewrite equation

\[ \hat{H} u = A(y, t) \chi_{\{u > 0\}} \]

where \( \hat{H} = H + \gamma (1 - \rho^2) a^2 g_y D_y \).
Proof

★ Rewrite equation

\[ \hat{H}u = A(y, t)\chi_{\{u>0\}} \]

where \( \hat{H} = H + \gamma(1 - \rho^2)a^2g_yD_y \).

★ Scale by \( r^3 \)

\[ u_r(y, t) = \frac{u(ry + \beta_0, r^2t)}{r^3} \]

and take the limit \( r \to 0 \)

\[ D_tu_0 - \frac{1}{2}a_0^2D_yu_0 = A_0y\chi_{\{u_0>0\}}. \]
Proof

★ Rewrite equation

\[ \hat{H} u = A(y, t) \chi \{ u > 0 \} \]

where \( \hat{H} = H + \gamma (1 - \rho^2) a^2 g_y D_y \).

★ Scale by \( r^3 \)

\[ u_r(y, t) = \frac{u(ry + \beta_0, r^2 t)}{r^3} \]

and take the limit \( r \to 0 \)

\[ D_t u_0 - \frac{1}{2} a_0^2 D_y^2 u_0 = A_0 y \chi \{ u_0 > 0 \} \cdot \]

★ Self-similar solution in the variable \( \xi = -y / \sqrt{t} \).

\( \tilde{u}(\xi) = u(y, t) \).

\[ -\tilde{u}'' - \frac{1}{2a_0^2} \xi \tilde{u}' + \frac{3}{2a_0^2} = -A_0 \xi \quad \text{in} \quad \{ \tilde{u} > 0 \} \]
Free boundary regularity: $\beta_0 = K$

**Theorem 2** There exists a modulus of continuity $\sigma(r)$ such that

$$(\beta(t), t) \in \{(y, t) : t < (y - K)^2 \sigma(y - K)\}.$$
Proof

Scale by $r$

\[ h_r(y, t) = \frac{h(ry + K, r^2 t)}{r} \]

and take limit $r \to 0$

\[
\min(D_t h_0 - \frac{1}{2} a_0^2 D_y^2 h_0, h_0 - g_0) = 0 \\
h_0(y, 0) = g_0(y)
\]
Proof

★ Scale by \( r \)

\[
h_r(y, t) = \frac{h(ry + K, r^2 t)}{r}
\]

and take limit \( r \to 0 \)

\[
\min(D_t h_0 - \frac{1}{2} a_0^2 D_y^2 h_0, h_0 - g_0) = 0
\]

\[
h_0(y, 0) = g_0(y)
\]

★ \( g_0 = y^+ \) is a strict subsolution to the limit PDE.

\[
(\beta(t), t) \in \{ t < (y - K)^2 \sigma(y - K) \}.
\]