Matrix Subordinators and Multivariate OU-based Volatility Models

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Synopsis

- Intro
- Volatility and OU processes
- Matrix subordinators
- Infinite divisibility in cones
- CLT for RMPV
- Positive definite matrix processes of OU type
- Roots of positive definite processes
Intro

Let $Y_t$ denote a $d$-dimensional vector of log prices, modelled as a Brownian semimartingale

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s$$

★ OU modelling of $\Sigma = \sigma^T \sigma$. One-dimensional case: realism and analytical tractability

★ Multipower Variation RMPV: Basis for inference on $\Sigma_t^+ = \int_0^t \Sigma_s ds$ where $\Sigma_s = \sigma_s^T \sigma_s$ and more generally on $\Sigma_t^{+r} = \int_0^t \Sigma_r ds$.

★ The MPV theory uses SDE representations of $d\sigma$ (not $d\Sigma$). Need SDE representations of $\Sigma^r$, in particular $\Sigma^{1/2}$
Volatility and OU processes

Univariate OU volatility

\[ d\sigma_t^2 = -\lambda \sigma_t^2 dt + dL_{\lambda t} \]

where \( \lambda > 0 \) is a parameter and \( L \) is a subordinator, i.e. a Lévy process with nonnegative increments.
The solution can be shown to be

\[ \sigma_t^2 = e^{-\lambda t} \sigma_0^2 + \int_0^t e^{-\lambda (t-s)} dL_s \lambda \]

Provided \( E(\log^+(L_t)) < \infty \) there is a unique stationary solution given by

\[ \sigma_t^2 = \int_{-\infty}^t e^{-\lambda (t-s)} dL_s \lambda \]
Volatility and OU processes

There is a vast literature concerning the extension of OU processes to \( \mathbb{R}^d \)-valued processes.

By identifying \( M_d \), the class of \( d \times d \) matrices, with \( \mathbb{R}^{d^2} \) one immediately obtains matrix valued processes.

So for a given Lévy process \( (L_t)_{t \in \mathbb{R}} \) with values in \( M_d \) and a linear operator \( A : M_d \to M_d \), a solution to the SDE

\[
dX_t = AX_{t-}dt + dL_t
\]

is termed a matrix-valued process of Ornstein-Uhlenbeck type.
As in the univariate case one can show that for some given initial value $X_0$ the solution is unique and given by

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}dL_s.$$ 

Provided $E(\log^+ ||L_t||) < \infty$ and $\sigma(A) \in (-\infty,0) + i\mathbb{R}$, there exists a unique stationary solution given by

$$X_t = \int_{-\infty}^t e^{A(t-s)}dL_s.$$
Matrix subordinators

However, in order to obtain positive semidefinite Ornstein-Uhlenbeck processes we need to consider matrix subordinators as driving Lévy processes.

Let $\bar{S}_d^+$ be the closure of the cone $S_d^+$ of positive definite matrices in $M_d$.

**Definition** A process $L$ with values in $\bar{S}_d^+$ and having independent stationary increments is called a *matrix subordinator*.
Infinite divisibility in the cone $\tilde{S}_d^+$

A random matrix $M$ is infinitely divisible in $\tilde{S}_d^+$ if and only if for each integer $p \geq 1$ there exist $p$ independent identically distributed random matrices $M_1, \ldots, M_p$ in $\tilde{S}_d^+$ such that $M \xrightarrow{law} M_1 + \ldots + M_p$.

**Lévy-Khintchine representation** (Skorohod (1991))

A random matrix $M \in \tilde{S}_d^+$ is infinitely divisible in $\tilde{S}_d^+$ if and only if its cumulant transform is of the form

$$ C(\Theta; M) = \text{itr}(\gamma \Theta) + \int_{\tilde{S}_d^+} (e^{\text{itr}(X\Theta)} - 1) \rho(dX), \quad \Theta \in S_d^+, $$

where $\gamma \in \tilde{S}_d^+$ is called the drift and the Lévy measure $\rho$ is such that $\rho(S_d^+ \setminus \tilde{S}_d^+) = 0$ and $\rho$ has order of singularity

$$ \int_{\tilde{S}_d^+} \min(1, \text{tr}(X)) \rho(dX) < \infty. $$
Infinite divisibility in the cone $\tilde{S}_d^+$

Lévy-Itô decomposition:
If $\{L_t\}$ is a matrix subordinator with the above Lévy-Khintchine representation then it has a Lévy-Itô decomposition

$$L_t = t\gamma + \int_0^t \int_{\tilde{S}_d^+ \setminus \{0\}} x\mu(ds, dx)$$

where $\gamma \in \tilde{S}_d^+$ is a deterministic drift and $\mu(ds, dx)$ a Poisson random measure on $\mathbb{R}^+ \times \tilde{S}_d^+$ with

$$E(\mu(ds, dx)) = \text{Leb}(ds)\nu(dx),$$

$\text{Leb}$ denoting the Lebesgue measure and $\nu$ the Lévy measure of $L_t$. 
Examples

- **Quadratic Covariation** of \(d\)-dimensional Lévy processes
- **Gamma type matrix distribution**  
  Lévy density:
  
  \[
  \frac{|\Sigma|^{-<d>}}{(\text{tr}(X\Sigma^{-1}))^d} e^{\text{tr}(-X\Sigma^{-1})}
  \]

  where \(<d> = (d + 1)/2\) and \([d] = (d + 1)d/2\).

  Kumulant transform:

  \[
  \mathcal{K}(\Theta, R) = \int_{S^+_d} \log(1 + \text{tr}(U\Sigma^{1/2}\Theta\Sigma^{1/2}))^{-1} dU.
  \]
Examples

★ **Bessel matrix distribution**  Lévy density:

\[
|\Sigma|^{-<d>} \int_{\forall > 0} \text{etr}(- \left\{ XY^{-1} + \Sigma^{-1} Y \right\}) \left( \text{tr}(Y\Sigma^{-1}) \right)^{-[d]-\beta} \frac{dY}{|Y|^{<d>}}.
\]

where **X** and **Y** are the anti-matrices of **X** and **Y**.
Interlude: CLT for RMPV

Central Limit Theory for Realised Multipower Variation
(B-N, Jacod, Graversen, Podolskij and Shephard (2006))

*Recall*: For a wide class of real–valued processes $Y$, including all semi-martingales, the *realised quadratic variation process*

$$V(Y; 2)_t^n = \sum_{i=1}^{[nt]} (Y_{i/n} - Y_{(i-1)/n})^2$$

converges in probability, as $n \to \infty$ and for all $t \geq 0$, towards the quadratic variation process $V(Y; 2)_t$ (usually denoted by $[Y, Y]_t$).
Interlude: CLT for RMPV

Next, let $r, s$ be nonnegative numbers. The realised bipower variation process of order $(r, s)$ is the increasing processes defined as:

$$V(Y; r, s)^n_t = n^{r+s-1/2} \sum_{i=1}^{nt} |Y_{i/n} - Y_{(i-1)/n}|^r |Y_{(i+1)/n} - Y_{i/n}|^s.$$  

Clearly $V(Y; 2)^n = V(Y; 2, 0)^n$.

The bipower variation process of order $(r, s)$ for $Y$, denoted by $V(Y; r, s)_t$, is the limit in probability, if it exists for all $t \geq 0$, of $V(Y; r, s)^n_t$.

**Uses:** Testing for jumps; Estimation of $\int_0^t \sigma_s^4 ds$ in the presence of jumps; ...
Interlude: CLT for RMPV

Extension to the multidimensional case.

Now $Y = (Y^j)_{1 \leq j \leq d}$ is taken as $d$–dimensional.

The \textit{realised cross–multipower variation processes} are defined by

$$V(Y^{j_1}, \ldots, Y^{j_N}; r_1, \ldots, r_N)_{nt}^n$$

$$= n^{\frac{r_1 + \ldots + r_N}{2} - 1} \sum_{i=1}^{\lfloor nt \rfloor} |Y_{i/n}^{j_1} - Y_{i-1/n}^{j_1}|^{r_1} \ldots |Y_{i+N-1/n}^{j_N} - Y_{i+N-2/n}^{j_N}|^{r_N}.$$
Interlude: CLT for RMPV

More generally still, let

$$X^n(g,h)_t = \frac{1}{n} \sum_{i=1}^{[nt]} g(\sqrt{n} \Delta_i^n Y) h(\sqrt{n} \Delta_{i+1}^n Y)$$

where $\Delta_i^n Y = Y_i^n - Y_{i-1}^n$, $g$ and $h$ are two maps on $R^d$, taking values in $\mathcal{M}_{d_1,d_2}$ and $\mathcal{M}_{d_2,d_3}$ respectively. So $X^n(g,h)_t$ takes its values in $\mathcal{M}_{d_1,d_3}$.

We refer to $X^n(g,h)$ as the realised multipower variation (RMPV) associated to $g$ and $h$. 
Interlude: CLT for RMPV

To derive a CLT for RMPV we need the following structural assumptions:

**Hypothesis (H):** We have

\[ \Upsilon_t = \Upsilon_0 + \int_0^t a_s ds + \int_0^t \sigma_s \ dW_s, \]

where \( W \) is a standard \( d' \)-dimensional BM, \( a \) is predictable \( R^d \)-valued locally bounded, and \( \sigma \) is \( \mathcal{M}_{d,d'} \)-valued càdlàg with \( \Sigma = \sigma \sigma^\top \) invertible.
Interlude: CLT for RMPV

Hypothesis (H’): We have

\[ \sigma_t = \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_s dW_s + \int_0^t v_s dV_s + \int_0^t \int_{E} \phi \circ w(s-, x)(\mu - v)(ds, dx) + \int_0^t \int_{E} (w - \phi \circ w)(s-, x) \mu(ds, dx). \]

where ****
Interlude: CLT for RMPV

**Hypothesis (K):** The function \( g \) and \( h \) are even and continuously differentiable, with partial derivatives having at most polynomial growth.

Now, recall that

\[
X^n(g, h)_t = \frac{1}{n} \sum_{i=1}^{[nt]} g(\sqrt{n} \Delta_i^n Y) h(\sqrt{n} \Delta_{i+1}^n Y)
\]

Under \((H), (H')\) and \((K)\), \( X^n(g, h) \) converges in probability to a process \( X(g, h) \).
Interlude: CLT for RMPV

**Theorem**  

**CLT for RMPV**  

Under \((H), (H')\) and \((K)\) the process

\[
\sqrt{n} \left( X_n(g, h) - X(g, h) \right)
\]

converges *stably in law* to the limiting process \(U(g, h)\) given componentwise by

\[
U(g, h)_{jk}^t = \sum_{j'=1}^{d_1} \sum_{k'=1}^{d_3} \int_0^t \alpha(\sigma_s, g, h)^{j'k'} \ dW_s^{j'k'}
\]

where \(W'\) is a multidimensional Brownian motion, independent of all the previous random objects, and where the coefficients \(\alpha(\sigma_s, g, h)\) satisfy ****.
Positive semidefinite matrix processes of OU type

\[ dX_t = AX_t \, dt + dL_t \]

**Proposition**  Let \( L_t \) be a matrix subordinator, assume that the linear operator \( A \) satisfies \( \exp(A t)(\bar{S}_d^+) \subseteq \bar{S}_d^+ \) for all \( t \in \mathbb{R}^+ \) and let \( X_0 \in \bar{S}_d^+ \).

Then the Ornstein-Uhlenbeck process \( (X_t)_{t \in \mathbb{R}^+} \) satisfying \( dX_t = AX_t \, dt + dL_t \) with initial value \( X_0 \) takes only values in \( \bar{S}_d^+ \).

If \( E(\log^+ \|L_t\|) < \infty \) and \( \sigma(A) \in (-\infty, 0) + i\mathbb{R} \), then the unique stationary solution \( (X_t)_{t \in \mathbb{R}} \) takes values in \( \bar{S}_d^+ \) only.
Positive semidefinite matrix processes of OU type

Which linear operators $A$ can one actually take to obtain both a unique stationary solution and ensure positive semidefiniteness?

The condition $\exp(A t) (S_d^+) \subseteq S_d^+$ means that for all $t \in \mathbb{R}^+$ the exponential operator $\exp(A t)$ has to preserve positive definiteness. So one needs to know first which linear operators on $S_d^+$ preserve positive definiteness.
Positive semidefinite matrix processes of OU type

Let $A : S_d \rightarrow S_d$ be a linear operator. Then $A(\tilde{S}_d^+) = \tilde{S}_d^+$, if and only if there exists a matrix $B \in GL_d$ such that $A$ can be represented as $X \mapsto BXB^*$.

Assume the operator $A : \tilde{S}_d^+ \rightarrow \tilde{S}_d^+$ is representable as $X \mapsto AX + XA^*$ for some $A \in M_d$. Then $e^{At}$ has the representation $X \mapsto e^{At}Xe^{A^*t}$ and $e^{At}(\tilde{S}_d^+) = \tilde{S}_d^+$ for all $t \in \mathbb{R}$.
Positive semidefinite matrix processes of OU type

For a linear operator $A$ of the latter type (i.e. $X \mapsto AX + XA^*$) the SDE for the OU process becomes

$$dX_t = (AX_t - X_t A^*)dt + dL_t$$

and the solution is

$$X_t = e^{At}X_0 e^{A^*t} + \int_0^t e^{A(t-s)}dL_s e^{A^*(t-s)}.$$
Positive semidefinite matrix processes of OU type

**Theorem** Let $(L_t)_{t \in \mathbb{R}}$ be a matrix subordinator with $E(\log^+ \|L_t\|) < \infty$ and let $A \in M_d$ such that $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}$.

Then the stochastic differential equation of Ornstein-Uhlenbeck type

$$dX_t = (AX_t^- + X_t^- A^*)dt + dL_t$$

has a unique stationary solution

$$X_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^*(t-s)}.$$

Moreover, $X_t \in \bar{S}^+_d$ for all $t \in \mathbb{R}$. 
Positive semidefinite matrix processes of OU type

Conditions ensuring that the stationary OU type process $X_t$ is almost surely strictly positive definite can be obtained:

**Theorem** If $\gamma \in S_d^+$ or $\nu(S_d^+) > 0$, then the stationary distribution $P_X$ of $X_t$ is concentrated on $S_d^+$. 
Positive semidefinite matrix processes of OU type

Extensive recent work by Christian Pigorsch, LMU, jointly with Robert Stelzer, TUM, on properties, extensions and applications of this general multivariate SV-OU framework.
Roots of positive semidefinite processes

To discuss the root questions we need a suitable *Itô formulae for finite variation processes in open sets*

**Definition**  Local Boundedness  Let \( (V, \| \cdot \|_V) \) be either \( \mathbb{R}^d, S^+_d \) or \( S_d \) with \( d \in \mathbb{N} \) and equipped with the norm \( \| \cdot \|_V \), let \( a \in V \) and let \( (X_t)_{t \in \mathbb{R}^+} \) be a \( V \)-valued stochastic process. We say that \( X_t \) is *locally bounded away from \( a \)* if there exists a sequence of stopping times \( (T_n)_{n \in \mathbb{N}} \) increasing to infinity almost surely and a real sequence \( (d_n)_{n \in \mathbb{N}} \) with \( d_n > 0 \) for all \( n \in \mathbb{N} \) such that \( \| X_t - a \|_V \geq d_n \) for all \( 0 \leq t < T_n \).

Likewise, we say for some open set \( C \subseteq V \) that the process \( X_t \) is *locally bounded within \( C \)* if there exists a sequence of stopping times \( (T_n)_{n \in \mathbb{N}} \) increasing to infinity almost surely and a sequence of compact convex subsets \( D_n \subseteq C \) with \( D_n \subseteq D_{n+1} \) for all \( n \in \mathbb{N} \) such that \( X_t \in D_n \) for all \( 0 \leq t < T_n \).
Roots of positive semidefinite processes

**Proposition**  *Itô formulae for finite variation processes in open sets*  
Let $(X_t)_{t \in \mathbb{R}^+}$ be a cadlag $\mathbb{R}^d$-valued process of finite variation (thus a semimartingale) with associated jump measure $\mu_X$ on $(\mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d \setminus \{0\}))$ and let $f : C \to \mathbb{R}^m$ be continuously differentiable, where $C \subseteq \mathbb{R}^d$ is an open set. Assume that the process $(X_t)_{t \in \mathbb{R}^+}$ is *locally bounded within* $C$. Then:
Roots of positive semidefinite processes

the process $X_t$ as well as its left limit process $X_{t-}$ take values in $C$ at all times $t \in \mathbb{R}^+$ and

$$f(X_t) = f(X_0) + \int_0^t Df(X_{s-})dX_s^c$$

$$+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (f(X_{s-} + x) - f(X_{s-})) \mu_X(ds, dx).$$
Univariate case

**Theorem** Let \((X_t)_{t \in \mathbb{R}^+}\) be a given adapted cadlag process which takes values in \(\mathbb{R}^+ \setminus \{0\}\), is locally bounded away from zero and can be represented as

\[
dX_t = c_t \, dt + \int_{\mathbb{R}^+ \setminus \{0\}} g(t -, x) \mu(dt, dx)
\]

where \(c_t\) is a predictable and locally bounded process, \(\mu\) a Poisson random measure on \(\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{0\}\) and \(g(s, x)\) is \(\mathcal{F}_s \times \mathcal{B}(\mathbb{R}^+ \setminus \{0\})\) measurable in \((\omega, x)\) and cadlag in \(s\). Moreover, \(g(s, x)\) takes only non-negative values. Then:
Roots of positive semidefinite processes

for any $0 < r < 1$ the unique positive process $Y_t = X_t^r$ is representable as

$$Y_0 = X_0^r, \quad dY_t = a_t dt + \int_{\mathbb{R}^+ \setminus \{0\}} w(t-, x) \mu(dt, dx),$$

where the drift

$$a_t := rX_{t-}^{r-1}c_t$$

is predictable and locally bounded and where

$$w(s, x) := (X_s + g(s, x))^r - (X_s)^r$$

is $\mathcal{F}_s \times \mathcal{B}(\mathbb{R}^+) \text{ measurable in } (\omega, x)$ and cadlag in $s$. Moreover, $w(s, x)$ takes only non-negative values.
Roots of positive semidefinite processes

When applied to subordinators this gives

**Corollary** Let \((L_t)_{t \in \mathbb{R}^+}\) be a Lévy subordinator with initial value \(L_0 \in \mathbb{R}^+\), associated drift \(\gamma\) and jump measure \(\mu\). Then for \(0 < r < 1\) we have that the unique positive process \(L_t^r\) is of finite variation and

\[
dL_t^r = r\gamma L_t^r \, dt + \int_{\mathbb{R}^+ \setminus \{0\}} ((L_t^- + x)^r - L_t^r) \, \mu(dt, dx),
\]

where the drift \(r\gamma L_t^r \) is predictable. Moreover, the drift is locally bounded if and only if \(L_0 > 0\) or \(\gamma = 0\).
Roots of positive semidefinite processes

**Multivariate case**  Generalisation of previous results:

**Theorem** Let $\left( X_t \right)_{t \in \mathbb{R}^+}$ be a given adapted cadlag process which takes values in $S_d^+$, is locally bounded within $S_d^+$ and can be represented as

$$dX_t = c_t dt + \int_{\tilde{S}_d^+ \setminus \{0\}} g(t-, x) \mu(dt, dx)$$

where $c_t$ is an $S_d^+$-valued, predictable and locally bounded process, $\mu$ a Poisson random measure on $\mathbb{R}^+ \times \tilde{S}_d^+ \setminus \{0\}$, and $g(s, x)$ is $\mathcal{F}_s \times \mathcal{B}(\tilde{S}_d^+ \setminus \{0\})$ measurable in $(\omega, x)$ and cadlag in $s$. Furthermore, $g(s, x)$ takes only values in $\tilde{S}_d^+$. Then
Roots of positive semidefinite processes

the unique positive definite square root process $Y_t = \sqrt{X_t}$ is given by

$$Y_0 = \sqrt{X_0}, \quad dY_t = a_t dt + \int_{S_d^+ \setminus \{0\}} w(t-, x) \mu(dt, dx),$$

with

$$a_t = X_{t-}^{-1} c_t,$$

where $X_{t-}$ is the linear operator $Z \mapsto \sqrt{X_{t-} Z} + Z \sqrt{X_{t-}}$ on $M_d$ and

$$w(s, x) := \sqrt{X_s + g(s, x)} - \sqrt{X_s}$$

Moreover, $w(s, x)$ takes only positive semidefinite values.
Corollary  Let \((L_t)_{t \in \mathbb{R}^+}\) be a matrix subordinator with initial value \(L_0 \in \mathcal{S}_d^+\), associated drift \(\gamma\) and jump measure \(\mu\). Then the unique positive semidefinite process \(\sqrt{L_t}\) is of finite variation and, provided that either \(L_0 \in S_d^+\) or \(\gamma \in S_d^+ \cup \{0\},\)

\[d\sqrt{L_t} = \mathbb{I}_{t-}^{-1}\gamma dt + \int_{\mathcal{S}_d^+ \setminus \{0\}} \left(\sqrt{L_{t-} + x} - \sqrt{L_{t-}}\right) \mu(dt, dx),\]

where \(\mathbb{I}_{t-}\) is the linear operator on \(M_d\) with \(Z \mapsto \sqrt{L_{t-}}Z + Z\sqrt{L_{t-}}\). The drift \(\mathbb{I}_{t-}^{-1}\gamma\) is predictable, and additionally locally bounded provided \(L_0 \in \mathcal{S}_d^+\) or \(\gamma = 0\).
Roots of Ornstein-Uhlenbeck processes

Finally we specialise to the behaviour of the roots of positive Ornstein-Uhlenbeck processes.

Recall that the driving Lévy process $L_t$ is assumed to be a (matrix) subordinator.

**Univariate case**

Let $X_t$ be a stationary process of OU type with driving Lévy subordinator $L_t$ (having non-zero Lévy measure) with a vanishing drift $\gamma$. Then for $0 < r < 1$ the stationary process $Y_t = X_t^r$ can be represented as

$$Y_t = \int_{-\infty}^{t} \int_{\mathbb{R}^+ \setminus \{0\}} e^{-\lambda r(t-s)} \left( (X_{s^-} + x)^r - X_{s^-}^r \right) \mu(ds, dx).$$
Roots of Ornstein-Uhlenbeck processes

Multivariate case

Proposition Let $X_t$ be a stationary process of OU type with driving matrix subordinator $L_t$ with a vanishing drift $\gamma$. Then the stationary process $Y_t = \sqrt{X_t}$ can be represented as

$$\int_{-\infty}^{t} \int_{S_d^+ \setminus \{0\}} \left( \sqrt{e^{A(t-s)}(X_s^- + x)e^{A^*(t-s)}} - \sqrt{e^{A(t-s)}X_s^-e^{A^*(t-s)}} \right) \mu(dx, ds)$$
References

