Bounds for Asian basket options

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1. Introduction: problem description and motivation

- Bounds for European-style discrete arithmetic Asian basket options in a Black & Scholes framework

- Consider a basket with $n$ assets whose prices $S_i(t)$, $i = 1, \ldots, n$, are described, under the risk neutral measure $\mathbb{Q}$ and with $r$ some risk-neutral interest rate, by

$$dS_i(t) = r S_i(t) dt + \sigma_i S_i(t) dW_i(t),$$

- Assume that the different asset prices are instantaneously correlated in a constant way i.e.

$$\text{corr}(dW_i, dW_j) = \rho_{ij} dt.$$ 

- Given the above dynamics, the $i$-th asset price at time $t$ equals

$$S_i(t) = S_i(0) e^{(r - \frac{1}{2} \sigma_i^2) t + \sigma_i W_i(t)}.$$
Price of a discrete arithmetic Asian basket call option with a fixed strike \( K \) and maturity \( T \) on \( m \) averaging dates at current time \( t = 0 \) is determined by

\[
ABC(n, m, K, T) = e^{-rT} \mathbb{E}^Q \left[ \left( \sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l(T - j) - K \right)_+ \right]
\]

with \( a_l \) and \( b_j \) positive coefficients, which both sum up to 1.

- For \( T \leq m - 1 \), the Asian basket call option is said to be in progress and for \( T > m - 1 \), we call it forward starting.

- Suitable for hedging as their payoff depend on an average of asset prices at different times and of different assets, and takes the correlations between the assets in the basket into account.

- No closed-form solutions available in the Black & Scholes setting.

- Dahl and Benth (2001a,b)\(^5\) value such options by quasi-Monte Carlo techniques and singular value decomposition.

- In the setting of Asian options, Thompson (1999a) used a first order approximation of the arithmetic sum and derived an upper bound that sharpens those of Rogers and Shi (1995)\(^6\).


Lord (2006) revised Thompson’s method and proposed a shift lognormal approximation to the sums and he included a supplementary parameter which is estimated by an optimization algorithm.

Deelstra et al. (2004) and Vanmaele et al. (2006) used techniques based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas et al. (2000)\textsuperscript{7} and Dhaene et al. (2002a)\textsuperscript{8}, combined with conditioning techniques and splitting up like in Curran (1994)\textsuperscript{9}, Rogers and Shi (1995) and Nielsen and Sandmann (2003)\textsuperscript{10}.

A random vector \((X^c_1, \ldots, X^c_k)\) is \textit{comonotonic} if each two possible outcomes \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) of \((X^c_1, \ldots, X^c_k)\) are ordered componentwise.

Consider \(S = \sum_{i=1}^k X_i\) with \(\underline{X} = (X_1, \ldots, X_k)\) with known marginal distributions but unknown dependence structure. Then the sum \(S\) is bounded below and above in convex order \((\preceq_{cx})\) by sums of comonotonic variables:

\[
S^\ell \preceq_{cx} S \preceq_{cx} S^u \preceq_{cx} S^c,
\]

which implies by definition of convex order that

$$
\mathbb{E}\left[(S^\ell - d)_+\right] \leq \mathbb{E}\left[(S - d)_+\right] \leq \mathbb{E}\left[(S^u - d)_+\right] \leq \mathbb{E}\left[(S^c - d)_+\right]
$$

for all $d$ in $\mathbb{R}^+$, while $\mathbb{E}\left[S^\ell\right] = \mathbb{E}\left[S\right] = \mathbb{E}\left[S^u\right] = \mathbb{E}\left[S^c\right]$. 
Write the double sum $S = \sum_{l=1}^{n} a_l \sum_{j=0}^{m-1} b_j S_l (T - j)$ as:

$$S = \sum_{i=1}^{mn} X_i = \sum_{i=1}^{mn} \alpha_i e^{Y_i}$$

with

$$\alpha_i = a_{\lceil \frac{i}{m} \rceil} b_{(i-1) \mod m} S_{\lceil \frac{i}{m} \rceil}(0) e^{(r-\frac{1}{2} \sigma^2_{Y_i}(T-(i-1) \mod m))}$$

and

$$Y_i = \sigma_{\lceil \frac{i}{m} \rceil} W_{\lceil \frac{i}{m} \rceil}(T - (i - 1) \mod m) \sim \mathcal{N}(0, \sigma^2_{Y_i} = \sigma^2_{\lceil \frac{i}{m} \rceil}(T - (i - 1) \mod m))$$

for all $i = 1, \ldots, mn$, where $\lceil x \rceil$ is the smallest integer greater than or equal to $x$ and

$$y \mod m = y - \lfloor y/m \rfloor m;$$

$\lfloor y \rfloor$ denotes the greatest integer less than or equal to $y$. 
Comonotonic Upper Bound

• Define $S^c = F^{-1}_{X_1}(U) + F^{-1}_{X_2}(U) + \cdots + F^{-1}_{X_{nm}}(U)$ with $U$ a Uniform$(0, 1)$ random variable and consider

$$ABC(n, m, K, T) \leq e^{-rT}E^Q [(S^c - K)_+]$$

• Comonotonic upper bound given by

$$ABC(n, m, K, T) \leq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_{l} b_{j} S_{l}(0) e^{-rj} \Phi \left[ \sigma_{l} \sqrt{T - j} - \Phi^{-1}(F_{S^c}(K)) \right]$$

$$- e^{-rT} K (1 - F_{S^c}(K))$$

where the cdf of the comonotonic sum $F_{S^c}(K)$ solves

$$\sum_{l=1}^{n} \sum_{j=0}^{m-1} a_{l} b_{j} S_{l}(0) \exp \left[ (r - \frac{1}{2} \sigma_{l}^2)(T - j) + \sigma_{l} \sqrt{T - j} \Phi^{-1}(F_{S^c}(K)) \right] = K$$

where $\Phi$ is the cumulative distribution function of a normally distributed random variable.
The payoff of the Asian basket call option satisfies

\[
\left( \sum_{l=1}^{n} a_l (S_l(T) - K_l) \right) + \left( \sum_{j=0}^{m-1} \sum_{l=1}^{n} b_j S_j(T) - K_j \right) \leq \left( \sum_{l=1}^{n} a_l (S_l(T) - K_l) \right) + \left( \sum_{j=0}^{m-1} \sum_{l=1}^{n} b_j S_j(T) - K_j \right) + \left( \sum_{l=1}^{n} a_l K_l = K \right) + \left( \sum_{j=0}^{m-1} b_j K_j \right).
\]

as well as

\[
\left( \sum_{l=1}^{n} a_l (S_l(T) - K_l) \right) + \left( \sum_{j=0}^{m-1} \sum_{l=1}^{n} b_j S_j(T) - K_j \right) \leq \left( \sum_{l=1}^{n} a_l (S_l(T) - K_l) \right) + \left( \sum_{j=0}^{m-1} \sum_{l=1}^{n} b_j S_j(T) - K_j \right) + \left( \sum_{l=1}^{n} a_l K_l = K \right) + \left( \sum_{j=0}^{m-1} b_j K_j \right).
\]
By a no-arbitrage argument we find that the time zero price of such Asian basket option should satisfy the following two relations:

\[ ABC(n, m, K, T) \leq \sum_{l=1}^{n} a_l AC_l(m, K_l, T) \leq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j e^{-r_j} C_l(K_{lj}, T - j) \]

\[ ABC(n, m, K, T) \leq \sum_{j=0}^{m-1} b_j e^{-r_j} BC(n, K_j, T - j) \leq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j e^{-r_j} C_l(K_{lj}, T - j). \]

Superreplication by a static portfolio of vanilla call options \( C_l \) on the underlying assets \( S_l \) in the basket and with different maturities and strikes.

Also an average of Asian options \( AC_l \) or a combination of basket options \( BC \) with different maturity dates form a superreplicating strategy.

Simon et al. (2000),
Albrecher et al. (2005)
Hobson et al. (2005) for a basket option in a model-free framework
Chen et al. (2007) for a class of exotic options in a model-free framework.
Comonotonic Lower Bound

• A lower bound, in the sense of convex order, for $S = \sum_{i=1}^{mn} X_i$ is

$$S^\ell = E[S \mid \Lambda]$$

where $\Lambda$ is a normally distributed random variable such that $(W_l(T - j), \Lambda)$ are bivariate normally distributed for all $l$ and $j$, which we assume in the sequel.

• Define $r_{l,j}$ by

$$r_{l,j} = \frac{\text{Cov}(W_l(T - j), \Lambda)}{\sigma_\Lambda \sqrt{T - j}}$$

• Assume that all $r_{l,j}$ are positive: then the comonotonic lower bound is determined by (and the case of all $r_{l,j}$ negative can be treated in a similar way):

$$ABC(n, m, K, T) \geq \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0)e^{-r_l j} \Phi \left[ r_{l,j} \sigma_l \sqrt{T - j} - \Phi^{-1}(F_{S^\ell}(K)) \right]$$

$$\quad - e^{-r T}K(1 - F_{S^\ell}(K))$$

where $F_{S^\ell}(K)$, solves

$$K = \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) \exp \left[ r_{l,j} \sigma_l \Phi^{-1}(F_{S^\ell}(K)) \sqrt{T - j} + \left( r - \frac{1}{2} r_{l,j}^2 \sigma_l^2 \right)(T - j) \right]$$
Choice of the Conditioning random variable

- In order to get a good lower bound, $\Lambda$ and $S$ should be as alike as possible

- take $\Lambda = FA1$ or $FA2$ such that for $i = 1, 2$:

$$FAi = \sum_{k=1}^{n} \sum_{p=0}^{m-1} a_kb_p c_i(k, p)\sigma_k S_k(0) W_k(T - p),$$

with

$$c_1(k, p) = e^{(r - \frac{1}{2} \sigma_k^2)(T - p)}, \quad c_2(k, p) = 1.$$

or as in Vanduffel et al. (2005)\textsuperscript{11} maximizing the first order approximation of the variance of $S^\ell$:

$$FA3 = \sum_{k=1}^{n} \sum_{j=0}^{m-1} a_kb_p S_k(0) e^{r(T - j)} \left[ (r - \frac{1}{2} \sigma_k^2) (T - j) + \sigma_k W_k(T - j) \right]$$

- Nielsen and Sandmann (2003) propose the standardized logarithm $GA$ of the geometric average $G$ which is defined by

$$G = \prod_{l=1}^{n} \prod_{j=0}^{m-1} S_l(T - j)^{a_l b_j} = \prod_{l=1}^{n} \left( \prod_{j=0}^{m-1} \left( S_l(0)e^{(r - \frac{1}{2} \sigma_l^2)(T - j) + \sigma_l W_l(T - j)} \right)^{b_j} \right)^{a_l}$$

Bounds based on the Rogers and Shi approach

- Rogers and Shi approach:

\[
ABC(n, m, K, T) \leq e^{-rT} \left\{ E^Q \left[ (S^\ell - K)_+ \right] + \frac{1}{2} E^Q \left[ \sqrt{\text{Var}(S | \Lambda)} \right] \right\}
\]

- Use idea of Nielsen and Sandmann (2003) and use reasoning of Deelstra et al. (2004):

\[
ABC(n, m, K, T) \leq e^{-rT} E^Q \left[ (S^\ell - K)_+ \right] + \frac{1}{2} e^{-rT} \int_{-\infty}^{d_\Lambda} \text{Var}(S | \Lambda = \lambda)^{\frac{1}{2}} f_\Lambda(\lambda) d\lambda
\]

where \( d_\Lambda \) is determined such that \( \Lambda \geq d_\Lambda \) implies that \( S \geq K \) and where \( f_\Lambda \) is the normal density function of \( \Lambda \).
The upper bound \( UBRS_{\Lambda} \) for the Asian basket option price follows:

\[
ABC(n, m, K, T) \leq e^{-rT} E^Q \left[ (S^\ell - K)_+ \right] + \frac{1}{2} e^{-rT} \left\{ \Phi \left( d^*_\Lambda \right) \right\}^{\frac{1}{2}}
\]

\[
\times \left\{ \sum_{l=1}^{n} \sum_{k=1}^{n} \sum_{j=0}^{m-1} \sum_{p=0}^{m-1} \sum a_l a_k b_j b_p S_l(0) S_k(0) \times e^{r(2T-j-p)} \left( e^{\sigma_l \sigma_k \rho_{lk} \min(T-j,T-p)} - e^{\sigma_l \sigma_k \rho_{lk} \sqrt{(T-j)(T-p)}} \right) \times \Phi \left( d^*_\Lambda - r_{l,j} \sigma_l \sqrt{T-j} - r_{k,p} \sigma_k \sqrt{T-p} \right) \right\}^{\frac{1}{2}}
\]

with \( d^*_\Lambda = \frac{d_{\Lambda} - E^Q[\Lambda]}{\sigma_\Lambda} \).

Until now, the lower bound was only known in a nice formula in the comonotonic situation, so also the upper bound \( UBRS_{\Lambda} \).
Partially Exact/Comonotonic Upper Bound

• For this upper bound a comonotonic situation is not necessary

• Write:

\[
E_Q[(S-K)_+] = E_Q[E_Q[(S-K)_+ | \Lambda]]
= E_Q[E_Q[(S-K)_+ | \Lambda] 1_{\Lambda \geq d_\Lambda}] + E_Q[E_Q[(S-K)_+ | \Lambda] 1_{\Lambda < d_\Lambda}]
\]

• Use

\[
E_Q[E_Q[(S-K)_+ | \Lambda] 1_{\Lambda < d_\Lambda}] \leq E_Q[E_Q[(S^u-K)_+ | \Lambda] 1_{\Lambda < d_\Lambda}].
\]

where \( S^u \) defined by \( S^u = F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \cdots + F_{X_{nm}|\Lambda}^{-1}(U) \) with \( U \) a Uniform(0, 1) random variable
and where \( d_\Lambda \) is determined such that \( \Lambda \geq d_\Lambda \) implies that \( S \geq K \).
We get the following expression

\[ABC(n, m, K, T)\]

\[\leq e^{-rT} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} \Phi \left( r_{l,j} \sigma_l \sqrt{T-j} - d^* \right) - e^{-rT} K (1 - \Phi(d^*)) \]

\[+ e^{-rT} \int_0^{\Phi(d^*)} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j) - \frac{1}{2} r_{l,j}^2 \sigma^2(T-j) + r_{l,j} \sigma_l \sqrt{T-j} \Phi^{-1}(v)} \]

\[\times \Phi \left( \sigma_l \sqrt{1 - r_{l,j}^2} (T-j) - \Phi^{-1}(F_{S^u|V=v}(K)) \right) dv -

\[K e^{-rT} \left( \Phi(d^*) - \int_0^{\Phi(d^*)} F_{S^u|V=v}(K) dv \right)\]

where \(F_{S^u|V=v}(K)\) follows from

\[K = \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r - \frac{1}{2} \sigma_l^2(T-j) + r_{l,j} \sigma_l \Phi^{-1}(v)) \sqrt{T-j} + \sigma_l \sqrt{(T-j)(1-r_{l,j}^2)} \Phi^{-1}(F_{S^u|V=v}(K))}.\]
3. Lower bound and upper bound UBRS in the Non-Comonotonic Case

- Not all $r_{l,j}$ have the same sign

- Write like suggested by Lord (2006) in the basket option case:

$$
\mathbb{E}^Q \left[ (S^\ell - K)_+ \right] = \mathbb{E}^Q \left[ (\mathbb{E}^Q [S | \Lambda] - K)_+ \right] \\
= \mathbb{E}^Q \left[ \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_{l} b_{j} S_l(0) e^{(r - \frac{1}{2} r_{l,j}^2 \sigma_l^2)(T-j) + r_{l,j} \sigma_l \sqrt{T-j}} - \frac{\Lambda - \mathbb{E}^Q[\Lambda]}{\sigma_{\Lambda}} - K \right] + \\
= \int_{0}^{1} \left[ \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_{l} b_{j} S_l(0) e^{(r - \frac{1}{2} r_{l,j}^2 \sigma_l^2)(T-j) + r_{l,j} \sigma_l \sqrt{T-j} \Phi^{-1}(v)} - K \right] dv
$$

- Denote $f(v) = \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_{l} b_{j} S_l(0) e^{(r - \frac{1}{2} r_{l,j}^2 \sigma_l^2)(T-j) + r_{l,j} \sigma_l \sqrt{T-j} \Phi^{-1}(v)} - K$.

- Denote $v^*$ the minimum of the function $f$. 
Figure 1: Possible shape of the integrand
• Case 1 of $f(v) \geq 0$ for all $v$:

$$E^Q \left[(S^\ell - K)_+\right] = E^Q[S] - K$$

$$= \sum_{l=1}^n \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} - K$$

• Case 2 of $f(v^*) < 0$:

Denote $d_{\Lambda_1}$ and $d_{\Lambda_2}$ the two solutions of the following equation in $x$:

$$\sum_{l=1}^n \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2} r_{l,j}^2 \sigma_l^2)(T-j) + r_{l,j} \sigma_l \sqrt{T-j} - \frac{E^Q[\Lambda]}{\sigma_{\Lambda}}} = K$$

with $d_{\Lambda_1} \leq d_{\Lambda_2}$.
Then:

\[
E^Q \left[ (S^\ell - K)_+ \right]
= \int_{-\infty}^{d_{\Lambda_1}} \left( \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}r^2 \sigma_l^2)(T-j)+r_l \sigma_l \sqrt{T-j} \Phi^{-1}(\nu)} - K \right) f_{\Lambda}(\lambda) d\lambda + \\
+ \int_{d_{\Lambda_2}}^{\infty} \left( \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{(r-\frac{1}{2}r^2 \sigma_l^2)(T-j)+r_l \sigma_l \sqrt{T-j} \Phi^{-1}(\nu)} - K \right) f_{\Lambda}(\lambda) d\lambda
\]

\[
= \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} \Phi \left( d_{\Lambda_1}^* - r_{l,j} \sigma_l \sqrt{T-j} \right) - K \Phi \left( d_{\Lambda_1}^* \right) + \\
+ \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j S_l(0) e^{r(T-j)} \Phi \left( r_{l,j} \sigma_l \sqrt{T-j} - d_{\Lambda_2}^* \right) - K \Phi \left( -d_{\Lambda_2}^* \right)
\]

where for \( i = 1, 2 \)

\[
d_{\Lambda_i}^* = \frac{d_{\Lambda_i} - E^Q[\Lambda]}{\sigma_{\Lambda}}.
\]

- In our numerical examples, the first integral appears to be negligible.

- This lower bound can be used in the Rogers & Shi approach. So UBRS is possible in a non-comonotonic situation.
4. Generalization of the upper bound of Thompson

• Let

\[ f_l(T - j) = \mu_l(T - j) + \bar{\sigma} \left[ \sigma_l W_l(T - j) - \sum_{i=1}^{n} \sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T - k) \right] \]

be a random function where \( \mu_l(T - j) \) and \( \bar{\sigma} \) are deterministic functions with

\[
\sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \mu_l(T - j) = 1.
\]

(4)

• Price of an Asian basket option is bounded by

\[
ABC(n, m, K, T) \leq e^{-rT} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \mathbb{E}^Q \left[ (S_l(T - j) - K f_l(T - j))_+ \right]
\]

(5)

• Thompson took \( \bar{\sigma} = 1 \), but we will optimize over \( \bar{\sigma} \)
Consider the Lagrangian:

\[ L(\lambda, \{\mu_l(T-j)\}, \sigma) = \sum_{l=1}^{n-1} \sum_{j=0}^{m-1} a_{b_j \mu_l} (T-j) - \lambda \left( \sum_{l=1}^{n-1} \sum_{j=0}^{m-1} a_{b_j \mu_l} (T-j) - 1 \right) \]

Consider the first order derivative with respect to \(\{\mu_l(T-j)\}\). Equating to zero leads to:

\[ [\sum_{l=1}^{n-1} \sum_{j=0}^{m-1} a_{b_j \mu_l} (T-j) - 1] = \frac{\lambda}{K} \]

Thompson suggests to approximate

\[ \exp(\sigma_i W_i (T-j)) \approx 1 + \sigma_i W_i (T-j) \]

valid for small \(\sigma_i \sqrt{T-j}\).
and to use as first order approximations for $S_l(T - j)$ and $Y_l(T - j)$ respectively:

\[
S_{l}^{FA}(T - j) = S_l(0)e^{(r - \frac{1}{2}\sigma_l^2)(T - j)}(1 + \sigma_l W_l(T - j))
\]

\[
Y_{l}^{FA}(T - j) = S_{l}^{FA}(T - j) - \bar{\sigma}K \left[ \sigma_l W_l(T - j) - \sum_{i=1}^{n} \sum_{k=0}^{m-1} a_i b_k \sigma_i W_i(T - k) \right].
\]

\[\cdot \text{ Denote } \mu_{l}^{FA}(T - j) \text{ by analogy to } \mu_l(T - j) \text{ and conclude that}\]

\[
Q \left[ Y_{l}^{FA}(T - j) \geq K \mu_{l}^{FA}(T - j) \right] = -\frac{\lambda}{K}
\]

Using the fact that $Y_{l}^{FA}(T - j)$ is normally distributed, we deduce that

\[
\frac{K \mu_{l}^{FA}(T - j) - S_l(0)e^{(r - \frac{1}{2}\sigma_l^2)(T - j)}}{\sqrt{\text{Var} \left( Y_{l}^{FA}(T - j) \right)}} = \gamma^{FA} = \Phi^{-1}(1 + \frac{\lambda}{K})
\]

and hence

\[
\mu_{l}^{FA}(T - j) = \frac{1}{K} \left( S_l(0)e^{(r - \frac{1}{2}\sigma_l^2)(T - j)} + \gamma^{FA} \sqrt{\text{Var} \left( Y_{l}^{FA}(T - j) \right)} \right)
\]

The constraint (4) for $\mu_{l}^{FA}(T - j)$ implies that

\[
\gamma^{FA} = \frac{K - \sum_{i=1}^{n} \sum_{j=0}^{m-1} a_i b_j S_l(0)e^{(r - \frac{1}{2}\sigma_l^2)(T - j)}}{\sum_{i=1}^{n} \sum_{j=0}^{m-1} a_i b_j \sqrt{\text{Var} \left( Y_{l}^{FA}(T - j) \right)}}.
\]

As the upper bound in (5) holds for any function $\mu_l(T - j)$ satisfying the above
It is well-known that conditioned on $W_l(T - j) = x \sqrt{T - j}$ is normally distributed.

Define $c_l^{FA}(T - j, x, \sigma)$ and $d_l^2(T - j, \sigma)$ as the corresponding conditional mean and variance.
The upper bound then becomes with $\phi(\cdot)$ denoting the density of the standard normal distribution and $Z$ having a $\mathcal{N}(0, 1)$ distribution:

$$ABC(n, m, K, T)$$

$$\leq e^{-rT} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty} \mathbb{E}^Q \left[ \left( c_{l}^{\text{FA}}(T - j, x, \overline{\sigma}) + d_l(T - j, \overline{\sigma})Z \right)^+ \right] \phi(x) dx$$

$$\leq e^{-rT} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty} \left\{ c_{l}^{\text{FA}}(T - j, x, \overline{\sigma})\Phi \left( \frac{c_{l}^{\text{FA}}(T - j, x, \overline{\sigma})}{d_l(T - j, \overline{\sigma})} \right) 

+ d_l(T - j, \overline{\sigma})\phi \left( \frac{c_{l}^{\text{FA}}(T - j, x, \overline{\sigma})}{d_l(T - j, \overline{\sigma})} \right) \right\} \phi(x) dx$$

Use the algorithm suggested by Lord(2006):

1. Calculate the upper bound using $\mu_{l}^{\text{FA}}(T - j)$ for three carefully chosen values of $\overline{\sigma}$
2. Fit a quadratic function in $\overline{\sigma}$ to these computed values
3. Determine the value of $\overline{\sigma}$ in which the upper bound attains its minimum
4. Recalculate the upper bound in the approximately optimal $\overline{\sigma}$
 Approximate \( Y_l(T - j) \) by a shifted lognormal random variable in the form:

\[
Y^{SLN}_l(T - j) = \alpha(l, T - j) + \exp[\theta(l, T - j) + \omega(l, T - j)Z_l]
\]

where \( \alpha(l, T - j) \), \( \theta(l, T - j) \) and \( \omega(l, T - j) \) are the shift, mean and volatility functions and \( Z_l \) is a standard normal distribution.

Determine the parameters by equating the first three moments of the shifted lognormal to those of the original random variable \( Y_l(T - j) \).

Use the same reasoning as before and conclude that

\[
\mathbb{Q}[Y^{SLN}_l(T - j) \geq K\mu^{SLN}_l(T - j)] = -\frac{\lambda}{K}
\]

Replace \( \mu^{FA}_l(T - j) \) by \( \mu^{SLN}_l(T - j) \).

Upper bound given by

\[
ABC(n, m, K, T) \leq e^{-rT} \sum_{l=1}^{n} \sum_{j=0}^{m-1} a_l b_j \int_{-\infty}^{\infty} \left\{ c^{SLN}_l(T - j, x, \sigma) \Phi \left( \frac{c^{SLN}_l(T - j, x, \sigma)}{d_l(T - j, \sigma)} \right) \right. \\
+ d_l(T - j, \sigma) \phi \left( \frac{c^{SLN}_l(T - j, x, \sigma)}{d_l(T - j, \sigma)} \right) \left\{ \phi(x) dx \right.
\]

Apply minimization algorithm as described before.
6. Numerical results

Notations:

- $\Lambda$ can be $FA_1$, $FA_2$, $FA_3$ or $GA$,
- $LBA$ for both the comonotonic lower bound and the non-comonotonic lower bound
- $PECUBA$ for partially exact/comonotonic upper bound,
- $UBRSA$ for upper bound based on the Rogers & Shi approach,
- $CUB$ for comonotonic upper bound
- $PECUB = \min(PECUBFA_1, PECUBFA_2, PECUBFA_3, PECUBGA)$,
- $UBRS = \min(UBRSFA_1, UBRSA_2, UBRSA_3, UBRSGA)$,
- $LB = \max(LBFA_1, LBFA_2, LBFA_3, LBGA)$,
- ThompUB for Thompson’s upper bound with $\bar{\sigma} = 1$,
- ThompUBquad, and SLNquad for upper bound based on the first order approximation and on the shift lognormal approximations, which use a numerical optimization algorithm to approximate the optimal scale $\bar{\sigma}$.

- The moneyness of the option is defined as

$$
\frac{K}{\sum_{i=1}^{n} \sum_{j=0}^{m-1} a_i b_j E_Q[S_i(T - j)]} - 1.
$$
Asian basket option

- Data from Beisser (2001)\textsuperscript{12}: Asian basket options with monthly averaging written on a fictitious chemistry-pharma basket that consists of the five German DAX stocks listed in the tables below
- The annual risk-free interest rate $r$ is equal to 6% 
- The averaging period of all options is five months and starts five month before maturity

\(FA_1\), \(FA_2\) and \(FA_3\) lead not to comonotonic lower bounds since the correlations \(r_{l,j}\) do not have the same sign

<table>
<thead>
<tr>
<th>stock</th>
<th>initial stock price</th>
<th>initial weight (in %)</th>
<th>volatility (in %)</th>
<th>dividend yield (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BASF</td>
<td>42.55</td>
<td>25</td>
<td>33.34</td>
<td>2.59</td>
</tr>
<tr>
<td>Bayer</td>
<td>48.21</td>
<td>20</td>
<td>31.13</td>
<td>2.63</td>
</tr>
<tr>
<td>Degussa-Hüls</td>
<td>34.30</td>
<td>30</td>
<td>33.27</td>
<td>3.32</td>
</tr>
<tr>
<td>FMC</td>
<td>100.00</td>
<td>10</td>
<td>35.12</td>
<td>0.69</td>
</tr>
<tr>
<td>Schering</td>
<td>66.19</td>
<td>15</td>
<td>36.36</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 1: Stock characteristics
<table>
<thead>
<tr>
<th></th>
<th>BASF</th>
<th>Bayer</th>
<th>Degussa-Hüls</th>
<th>FMC</th>
<th>Schering</th>
</tr>
</thead>
<tbody>
<tr>
<td>BASF</td>
<td>1.00</td>
<td>0.84</td>
<td>−0.07</td>
<td>0.45</td>
<td>0.43</td>
</tr>
<tr>
<td>Bayer</td>
<td>0.84</td>
<td>1.00</td>
<td>0.08</td>
<td>0.62</td>
<td>0.57</td>
</tr>
<tr>
<td>Degussa-Hüls</td>
<td>−0.07</td>
<td>0.08</td>
<td>1.00</td>
<td>−0.54</td>
<td>−0.59</td>
</tr>
<tr>
<td>FMC</td>
<td>0.45</td>
<td>0.62</td>
<td>−0.54</td>
<td>1.00</td>
<td>0.86</td>
</tr>
<tr>
<td>Schering</td>
<td>0.43</td>
<td>0.57</td>
<td>−0.59</td>
<td>0.86</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 2: Correlation structure
Figure 2: Comparison of option values with $T = 0.5$

Figure 3: Comparison of option values with $T = 1$
Figure 4: Comparison of option values with $T = 5$

Figure 5: Comparison of option values with $T = 10$
<table>
<thead>
<tr>
<th>$T$ (moneyness)</th>
<th>$K$</th>
<th>To compare</th>
<th>Thomp UB</th>
<th>Thomp UBquad</th>
<th>SLN quad</th>
<th>LBA</th>
<th>UBRSA</th>
<th>PECUBA</th>
<th>$\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>40</td>
<td>MC:10.8465</td>
<td>10.8582</td>
<td>10.8580</td>
<td>10.8520</td>
<td>10.8448</td>
<td>10.8556</td>
<td>10.9042</td>
<td>FA1</td>
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<tr>
<td></td>
<td></td>
<td>(−0.2181)</td>
<td>SE:0.0057</td>
<td></td>
<td></td>
<td>10.8448</td>
<td>10.8558</td>
<td>10.9054</td>
<td>FA2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>CUB:11.1221</td>
<td></td>
<td></td>
<td>10.8448</td>
<td>10.8554</td>
<td>10.9023</td>
<td>FA3</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>MC:2.7860</td>
<td>2.9564</td>
<td>2.9442</td>
<td>2.9415</td>
<td>2.7801</td>
<td>2.8937</td>
<td>3.4912</td>
<td>FA1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(−0.0227)</td>
<td>SE:0.0040</td>
<td></td>
<td></td>
<td>2.7801</td>
<td>2.8930</td>
<td>3.4919</td>
<td>FA2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>CUB:4.3465</td>
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<td></td>
<td>2.7800</td>
<td>2.8903</td>
<td>3.4376</td>
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</tr>
<tr>
<td></td>
<td>60</td>
<td>MC:0.2338</td>
<td>0.3591</td>
<td>0.3417</td>
<td>0.3361</td>
<td>0.2299</td>
<td>0.4617</td>
<td>0.9288</td>
<td>FA1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1728)</td>
<td>SE:0.0012</td>
<td></td>
<td></td>
<td>0.2299</td>
<td>0.4613</td>
<td>0.9272</td>
<td>FA2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>CUB:1.1856</td>
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<td></td>
<td>0.2300</td>
<td>0.4573</td>
<td>0.9080</td>
<td>FA3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1742</td>
<td>1.1034</td>
<td>1.0407</td>
<td>GA</td>
</tr>
</tbody>
</table>
### Table 4: Valuation results for Asian basket call option

<table>
<thead>
<tr>
<th>$K$ (moneyness)</th>
<th>$T$ To compare</th>
<th>Thomp UB</th>
<th>Thomp UBquad</th>
<th>SLN quad</th>
<th>LBA</th>
<th>UBRSA</th>
<th>PECUBA</th>
<th>$\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>40 MC:17.3030</td>
<td>17.6536</td>
<td>17.6361</td>
<td>17.6159</td>
<td>16.9863</td>
<td>18.1608</td>
<td>18.5893</td>
<td>$FA_1$</td>
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<tr>
<td></td>
<td>($-0.3346$) SE:0.1319</td>
<td>17.0030</td>
<td>18.1698</td>
<td>18.6527</td>
<td>$FA_2$</td>
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<tr>
<td></td>
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<td>16.9727</td>
<td>18.1300</td>
<td>18.5112</td>
<td>$FA_3$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>16.9010</td>
<td>18.5126</td>
<td>18.8484</td>
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<tr>
<td></td>
<td>($-0.1807$) SE:0.0295</td>
<td>12.2421</td>
<td>13.8929</td>
<td>14.5678</td>
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<tr>
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<td>11.9023</td>
<td>14.6519</td>
<td>14.9816</td>
<td>$GA$</td>
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<tr>
<td>60</td>
<td>60 MC:9.1299</td>
<td>10.0549</td>
<td>9.8545</td>
<td>9.8331</td>
<td>8.7834</td>
<td>11.0153</td>
<td>11.6879</td>
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<td>($-0.0168$) SE:0.0268</td>
<td>8.7774</td>
<td>10.9485</td>
<td>11.6439</td>
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<td>8.7853</td>
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<td>11.0728</td>
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<tr>
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<td>8.2379</td>
<td>12.0024</td>
<td>12.0553</td>
<td>$GA$</td>
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</tr>
<tr>
<td></td>
<td>$(0.1470)$ SE:0.0241</td>
<td>6.3127</td>
<td>9.0051</td>
<td>9.5048</td>
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<td>6.3376</td>
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<td>5.6654</td>
<td>10.2258</td>
<td>9.7925</td>
<td>$GA$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Results

- Non-comonotonic lower bounds $LB_{FA1}$, $LB_{FA2}$ and $LB_{FA3}$ perform better than the comonotonic lower bound $LB_{GA}$
- For short maturities and in- and at-the-money, UBRS outperforms all the other upper bounds
- Other cases: ThompUBquad and SLNquad provide the best upper bounds
- Lower bound is very close to the Monte Carlo value but looses a bit of its sharpness for larger maturities
- Precision of ThompUBquad and SLNquad decreases with the maturity $T$
- PECUB is only useful far-out-of-the-money, otherwise too high to be useful in comparison with ThompUBquad and SLNquad
7. Conclusion

- Bounds for the price of a discrete arithmetic Asian basket call option with fixed strike and this in the Black and Scholes setting.
- Very good lower bound in all cases: not only in the comonotonic case but also in the non-comonotonic case, which can then be applied to obtain the UBRS upper bound.
- UBRS is the best upper bound for short maturities and in- and at-the-money.
- SLNquad is the best upper bound for all other cases.

THANK YOU FOR YOUR ATTENTION!!!