Adaptive integration for multi-factor portfolio credit loss models

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1 Joint work with Cornelis W. Oosterlee at TU Delft and CWI.
OUTLINE

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   - Adaptive Genz-Malik Rule
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A credit portfolio consisting of $n$ obligors with exposure $w_1, w_2, \ldots, w_n$.

Default indicator $D_i = 1\{X_i < \gamma_i\}$, $X_i$ standardized log asset value and $\gamma_i$ default threshold.

Default probability $p_i = P(X_i < \gamma_i)$.

Portfolio loss $L = \sum_{i=1}^{n} w_i D_i$.

Tail probability $P(L > x)$ for some extreme loss level $x$.

Value at Risk (VaR): the $\alpha$-quantile of the loss distribution of $L$ for some $\alpha$ close to 1.
Latent factor models

\[ X_i = \alpha_{i1} Y_1 + \cdots + \alpha_{id} Y_d + \beta_i Z_i = \alpha_i Y^d + \beta_i Z_i, \]

- \( Y_1 \ldots Y_d \): systematic factors that affect more than one obligor, e.g., state of economy, effects of different industries and geographical regions.
- \( Z_i \): idiosyncratic factor that only affects an obligor itself.
- \( Y^d \) and \( Z_i \) are independent for all \( i \).
- \( D_i (Y^d) \) and \( D_j (Y^d) \) are independent.
- \( L (Y^d) = \sum w_i D_i (Y^d) \) becomes a weighted sum of independent Bernoulli random variables.
TAIL PROBABILITY: A NUMERICAL INTEGRATION PROBLEM

\[ P(L > x) = \int P(L > x | Y^d) \, dP(Y^d) \]

The integrand \( P(L > x | Y^d) \) can be calculated accurately by

- the saddlepoint approximation - Martin et al (2001a, b), Huang et al (2007a)
PROPERTIES OF THE CONDITIONAL TAIL PROBABILITY

Assuming the factor loadings, $\alpha_{ik}, i = 1, \ldots, n, k = 1, \ldots, d$ are all nonnegative,

- The mapping

$$ y_k \rightarrow P(L > x | Y_1 = y_1, Y_2 = y_2, \ldots, Y_d = y_d), \quad k = 1, \ldots, d $$

is non-increasing in $y_k$.

$$ \Downarrow $$

$$ \forall Y^d \in [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d] $$

$$ P(L > x | b_1, \ldots b_d) \leq P \left( L > x | Y^d \right) \leq P(L > x | a_1, \ldots a_d) $$

- $P(L > x | Y_1, Y_2, \ldots, Y_d)$ is continuous and differentiable with respect to $Y_k, k = 1, \ldots, d$. 

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A Gaussian one-factor example

**Figure:** The integrand $P(L > 100|Y)$ as a function of the common factor $Y$ for portfolio A, which consists of 1000 obligors with $w_i = 1$, $p_i = 0.0033$ and $\alpha_i = \sqrt{0.2}$, $i = 1, \ldots, 1000$. 
The Gaussian multi-factor model

\[ I(f) = P(L > x) = \int \cdots \int f(Y_1, \ldots Y_d) \phi(Y_1, \ldots Y_d) \, dY_1 \cdots dY_d, \]

where \( f(Y_1, \ldots Y_d) = P(L > x \mid Y_1, \ldots Y_d). \)

- curse of dimensionality: The product quadrature rule becomes impractical because the number of function evaluations grows exponentially with \( d \).
- (quasi-) Monte Carlo methods: sample uniformly in the cube \([0, 1]^d\).
- focus on the subregions where the integrand is most irregular \( \Rightarrow \) adaptive integration.
GLOBALLY ADAPTIVE ALGORITHMS

integration rule
error estimate
1. Choose a subregion from a collection of subregions and subdivide the chosen subregion.

2. Apply an integration rule to the resulting new subregions; update the collection of subregions.

3. Update the global integral and error estimate; check whether a predefined termination criterion is met; if not, go back to step 1.
The Genz-Malik rule

- A polynomial interpolatory rule of degree 7, which integrates exactly all monomials $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_d}$ with $\sum k_i \leq 7$ and fails to integrate exactly at least one monomial of degree 8.
- All integration nodes are inside integration domain.
- Requires $2^d + 2d^2 + 2d + 1$ integrand evaluations for a function of $d$ variables, most advantageous for problems with $d \leq 8$. Gauss-Legendre quadrature of degree 7 would need $4^d$ integrand evaluations.
- A degree 5 rule embedded in the degree 7 rule is used for error estimation, no additional integrand evaluations are necessary.

$$\epsilon = |l_7 - l_5|.$$
The Genz-Malik rule

- Bounded integral in each subregion.
  \( \forall \mathbf{Y}^d \in [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d] \)

  \[
  L \leq P \left( L > x \mid \mathbf{Y}^d \right) \leq U \Rightarrow \\
  L \prod_{i=1}^{d} (\Phi(b_i) - \Phi(a_i)) \leq I(f) \leq U \prod_{i=1}^{d} (\Phi(b_i) - \Phi(a_i)).
  \]

- Local bounds aggregate to a global upper bound and a global lower bound for the whole integration region.

- Asymptotic convergence: \( I_7 \rightarrow I(f) \) if we continue with the subdivision until the global upper bound and lower bound coincide.

Adaptive Monte Carlo integration

- Globally adaptive algorithm using Monte Carlo simulation as a basic integration rule.
- Asymptotically convergent.
- Unbiased estimate for the tail probability.
- Practical variance estimate, probabilistic error bounds available.
- Error convergence rate at worst $O\left(\frac{1}{\sqrt{N}}\right)$.
- Number of sampling points in each subregion independent of number of dimensions $d$. 
A 2D example

**Figure:** Adaptive Genz-Malik rule for a 2 factor model. (left) integrand \( P(L > x | Y_1, Y_2) \); (right) centers of the subregions generated by adaptive integration.

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A five-factor model

- 1000 obligors with $w_i = 1$, $p_i = 0.0033$, $i = 1, \ldots, 1000$, grouped into 5 buckets of 200 obligors.

- Factor loadings

\[
\alpha_i = \begin{cases} 
\left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), & i = 1, \ldots, 200, \\
\left( \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right), & i = 201, \ldots, 400, \\
\left( \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, 0, 0 \right), & i = 401, \ldots, 600, \\
\left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, 0, 0 \right), & i = 601, \ldots, 800, \\
\left( \frac{1}{\sqrt{2}}, 0, 0, 0, 0 \right), & i = 800, \ldots, 1000.
\end{cases}
\]

- Benchmark: plain MC with hundreds of millions of scenarios.
A five-factor model: Adaptive GM

**Figure:** Estimation relative errors of adaptive GM, plain MC and quasi-MC methods with around $N = 10^6$ evaluations for various loss levels.
A five-factor model: Adaptive MC

**Figure:** Tail probability $P(L > 400)$ computed by adaptive MC integration with number of integrand evaluations ranging from 50,000 to $10^6$ and their corresponding 95% confidence intervals (dotted lines). The dashed line is our Benchmark.
**A five-factor model: Adaptive MC**

**Figure**: Relative estimation error of $P(L > x)$ by Adaptive MC and plain MC for different loss levels $x$.
CONCLUSIONS

For the calculation of the tail probability in multi-factor portfolio credit loss models,

- Adaptive algorithms are very suitable and particularly attractive for large loss levels.
- Both adaptive Genz-Malik rule and adaptive Monte Carlo integration are asymptotically convergent.
- The adaptive Monte Carlo integration is able to provide practical probabilistic error bounds, with error convergence rate at worst $O\left(\frac{1}{\sqrt{N}}\right)$.  

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