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MULTIDIMENSIONAL COHERENT RISK MEASURES

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COHERENT RISK MEASURES

**Definition 1.2. (ADEH97)** A coherent utility function on $L^\infty$ is a map $u : L^\infty \to \mathbb{R}$ with the properties

(i) (Superadditivity) $u(X + Y) \geq u(X) + u(Y)$;

(ii) (Monotonicity) If $X \leq Y$, then $u(X) \leq u(Y)$;

(iii) (Translation invariance) $u(X + m) = u(X) + m$ for all $m \in \mathbb{R}$;

(iv) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \geq 0$;

(v) (Fatou property) If $\lim_{n} |X_n| \leq 1$, $X_n \xrightarrow{P} X$, then $u(X) \geq \lim_{n} u(X_n)$.

The corresponding coherent risk measure is $\rho(X) = -u(X)$. 
PARTIAL ORDERING

$X \prec Y$ if $X(\omega) - Y(\omega) \in K(\omega)$ P-a.s., where $K(\omega)$ is the cone of currency exchange rates.
**Definition 1.1.** A multidimensional coherent utility function on $(L^\infty)^d$ is a map $u: (L^\infty)^d \to \mathbb{C}$, $u \neq 0^d$, $u = u^+ + 0^d$ with the properties

(i) **(Superadditivity)** $u(X + Y) \geq u(X) + u(Y)$;

(ii) **(Monotonicity)** If $X \prec Y$, then $u(X) \subseteq u(Y)$;

(iii) **(Translation invariance)** $u(X + m) = u(X) + m$ for all $m \in \mathbb{R}^d$;

(iv) **(Positive homogeneity)** $u(\lambda X) = \lambda u(X)$ for $\lambda > 0$;

(v) **(Fatou property)** If $\|X_n\| \leq c$, $X_n \xrightarrow{p} X$, then $u(X) \geq \limsup_n u(X_n)$, i.e. if $x$ belongs to infinitely many $u(X_n)$, then $x$ belongs to $u(X)$.

The corresponding multidimensional coherent risk measure is $\rho(X) = -u(X)$. 
AXIOMS

Remarks.

(i) If $u$ is a coherent utility function, then $v(X) = (-\infty, u(X)]$ is a multidimensional coherent utility function with $d = 1$, $K(\omega) = \emptyset$.

(ii) If $u$ is a multidimensional coherent utility function with $d = 1$, $K(\omega) = \emptyset$, then $v(X) = \sup\{x \in \emptyset : x \in u(X)\}$ is a coherent utility function.
**Theorem 2.1.** A function $u : (L^\infty)^d \rightarrow C$ is a multidimensional coherent utility function iff there exists nonempty set $D \subseteq (L^1)^d$ such that $Z(\omega) \in K^*(\omega)$ P-a.s. and

$$u(X) = \left\{ x \in R^d : \forall Z \in D \sum_{i=1}^{d} EX^iZ^i \leq \sum_{i=1}^{d} E^iX^i \right\},$$

(2.1)

where $K^*(\omega)$ - polar to $K(\omega)$, i.e.

$$K^*(\omega) = \left\{ x \in R^d : \forall z \in K(\omega) \langle x, z \rangle \leq 0 \right\}.$$

**Theorem 2.2.** (ADEH99) A function $u : L^\infty \rightarrow \mathbb{R}$ is a coherent utility function iff there exists nonempty set $D \subseteq \mathcal{P}$ such that

$$u(X) = \inf_{Q \in D} E_{Q} X.$$  

(2.2)
Definition 2.3. The largest set, for which (2.1) is true, is called the **determining set** for multidimensional coherent utility function.

Definition 2.4. A multidimensional coherent utility function on \((L^0)^d\) is a map \(u: (L^0)^d \to C \cup \{\emptyset\}\) defined as

\[
u(X) = \left\{ x \in R^d : \forall Z \in D \sum_{i=1}^d E\left|X^iZ^i\right| \leq \sum_{i=1}^d E(X^iZ^i) \right\}, \tag{2.3}\]

where \(D\) – set of \(d\)-dimensional random vectors \(Z \in (L^1)^d\) such that \(Z(\omega) \in K^*(\omega)\) \(P\)-a.s., and \(E(X^iZ^i) = E(X^iZ^i)^+ - E(X^iZ^i)^-\), with an agreement \((+\infty) - (-\infty) = -\infty\), and if in the sum we have one item equal to \(-\infty\), then the sum is equal to \(-\infty\).
Remarks.

(i) The above definitions are the multidimensional analogues of the classical ones.

(ii) Clearly, the determining set is a convex cone. If multidimensional coherent utility function is on \((L^\infty)^d\), then the determining set is \((L^1)^d\)-closed.

(iii) If \(D = (L^1)^d\)-closed convex cone and multidimensional coherent utility function is defined by (2.1) or (2.3), then \(D\) is the determining set for \(u\).
Let $L = \left\{ Z \in (L^1)^d : \sum_{i=1}^{d} EZ^i = 1 \right\}$.

Then we can introduce some important spaces:

$$L^1_w(D) = \left\{ X \in (L^0)^d : \sup_{Z \in D \cap L} \sum_{i=1}^{d} |EZ^iX^i| < \infty \right\};$$

$$L^1_s(D) = \left\{ X \in (L^0)^d : \sup_{Z \in D \cap L} \sum_{i=1}^{d} |EZ^iX^i| I\{ |X_i| > n \} > 0 \right\}.$$
EXTREME ELEMENTS

**Definition 3.1.** Let $u$ be a multidimensional coherent utility function with the determining set $D$. Let $X \in (L^0)^d$, $x \in \partial u(X)$. We will call a nonzero vector $Z \in D$ an extreme element for $X$ at point $x$ if $\sum_{i=1}^{d} EZ^i X^i = \sum_{i=1}^{d} EZ^i x^i$.

The set of extreme elements for $X$ at point $x$ is denoted by $\chi_D(X,x)$.

**Proposition 3.2.** If $D \cap L$ is weakly compact and $X \in L^1_s(D)$, then $\chi_D(X,x) \neq \emptyset$. 
EXTREME ELEMENTS

The financial problems, for solution of which we use extreme elements:

(i) Capital allocation;

(ii) Risk contribution.
THANK YOU FOR YOUR ATTENTION!

- Axiomatization of multidimensional coherent risk measures (utility functions) and their connection with coherent risk measures (utility functions) in one-dimensional case.

- Representation of multidimensional coherent utility functions and introducing of the notion determining set in multidimensional case.

- Extreme elements as one of the basic objects for solving some financial problems.