A New Tool For Correlation Risk Management:
The Market Implied Comonotonicity Gap

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Main aims of this contribution

- Part I: Optimal arbitrage free static hedging strategies for basket options and new measure of lack of comonotonic or antimonotonic dependence in correlated assets: Market Implied Comonotonicity Gap (Joint work with Tai-Ho Wang, building on earlier work by Hobson, Laurence and Wang).

- Part II: Extension to generalized spread options.
We introduce a quantity called "the Gap", or more precisely "Market Implied Comonotonicity Gap" (for short: MICG), with the property that:

- **Gap** can be monitored over time and used as a tool in a static (or semi-static) dispersion trading strategy.

- When gap is small ("High correlation") compared to it's historical values: basket (consider case of index option first, later in talk spread) is overpriced.
  
  ⇒ Sell basket option, buy options on the components.

- When gap is big compared to it's historical values ("Low correlation"): basket is cheap, undervalued.

  ⇒ Buy an option on the basket, sell options on the components.
This is not an arbitrage strategy:

- It carries some risk, but downside risk is quite small.

- It is important to find the right time to enter into a "Gap Trade".
We will describe MICG and contrast with another well known dispersion trading strategy, so called "implied correlation."

**Implied correlation** is the number $\rho$ such that when $\rho_{ij}$ are replaced by $\rho$ gives same implied variance of index:

$$\sigma^2_I = \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{n} \sigma_i^2 + \rho \sum_{i \neq j} \sigma_i \sigma_j$$

Hence,

$$\rho = \frac{\sigma^2_I - \sum_{i=1}^{n} \sigma_i^2}{\sum_{i \neq j} \sigma_i \sigma_j}$$
But

\[ \sigma_I = \sigma_I(K^{\text{bask}}), \]

so which strikes \( K_i, i = 1, \ldots, n \) should we use to select \( \sigma_i = \sigma_i(K_i), i = 1, \ldots, n \) in the above formula?

Wide spread practice:

\[ K^{\text{bask}} \text{ ATM, then choose } K_i \text{ ATM} \]

But what if \( K^{\text{bask}} \) is out of or in the money? Or even for ATM in what sense is choice of ATM \( K_i \) optimal?

In contrast MICG gives means of selecting optimal strikes.
A new measure of correlation

Plan: We will recall the definition of comonotonicity and will illustrate the difference between perfect positive correlation and co-monotonicity.

We introduce as a measure of lack of comonotonicity of components in a basket product:

\[ \text{Gap} = C - M \]

\( C \): the market implied comonotonic price
\( M \): true market price
Recall the definition of comonotonicity:

A random vector \((X_1, X_2, \cdots, X_n)\) is said to be co-monotonic if there exists a uniformly distributed random variable \(U\) such that

\[
U \sim \text{Uniform}(0, 1)
\]

\[
(X_1, X_2, \cdots, X_n) \overset{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)),
\]

where \(F_{X_i}(x)\) is the distribution function of \(X_i\).
**Perfect positive correlation ≠ co-monotonicity**

**Difference** between perfect positive correlation and co-monotonicity. Tchen, Dhaene-Denuit’s theorem, concerning the relation of linear correlation with comonotonicity:

**Theorem 1** If \((X_1, X_2)\) is a random vector with given margins \(F_{X_1}, F_{X_2}\) and let \(\rho\) be the Pearson (i.e., linear, standard) correlation coefficient, then we have

\[
\rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1 - U)) \leq \rho(X_1, Y_1) \leq \rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)),
\]

where \(U\) is a uniformly distributed random variable.

In words:

- Largest value of the correlation for a random vector \((X_1, X_2)\) with given marginals is attained for comonotonic random variables, but is generally not equal to 1 unless they have a linear dependence with positive slope \((X_2 = aX_1 + b, a > 0)\).

- Minimal value of the correlation for a random vector \((X, X_2)\) with given marginals is attained for antimonotonic random variables, but is generally not equal to \(-1\).
Does the market offer a comononotonic Index?

- The answer of course is no.
- But, surprisingly, perhaps, we may synthetically create an index option that behaves "as if" the underlying assets were comonotonic.
- This synthetic comononotonic index option can be created using traded options on the individual components of the index, with judiciously chosen strikes.
were comonotonic
Continuous co-monotonic

Support of bivariate comonotonic distribution

\( S_2 \) is non-decreasing function of \( S_1 \)
Comonotonic Distribution: purely atomic, with jumps

A comonotonic distribution with jumps
How to determine $C$?

So, given a basket options with payoff

$$\left( \sum w_i S_i - K \right)^+$$

how do we determine the comonotonic price?

### ANSWER:
- **If** we knew with certainty the marginals $F_{S_i}$ of the individual assets $S_i$ in the basket, the procedure would be:
- **First** determine the joint probability distribution for the stocks in the basket via

$$P(S_1 \leq x_1, S_2 \leq x_2, \ldots, S_n \leq x_n) = C^U_{\text{Fréchet}}(F_{S_1}(x_1), F_{S_2}(x_2), \ldots, F_{S_n}(x_n))$$

where

$$C^U_{\text{Fréchet}}(y_1, y_2, \ldots, y_n) = \min(y_1, y_2, \ldots, y_n) \quad \text{upper Fréchet bound}$$
Second: Determine the density of joint prob. distribution of the basket via

\[
p(x_1, x_2, \cdots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} \left[ P(S_1 \leq x_1, S_2 \leq x_2, \cdots, S_n \leq x_n) \right]
\]

Third:

\[
Basket\ Price = \int_{\mathbb{R}_n^+} \left( \sum_{i=1}^{n} S_i - K \right)^+ p(S_1, S_2, \ldots, S_n) dS_1 \ldots dS_n
\]
Where do marginals come from?


**Theorem 2**  Let $C(S, t, K, T)$ be call prices corresponding at time $t$ and given that the spot price is at $S$, for a call option struck at $K$ and expiring at $T$, assuming a continuum of strikes is traded. Then

$$
\frac{\partial^2}{\partial K^2} C(S, t, K, T) = e^{-r(T-t)p(S, t, K, T)} \text{ where } p \text{ is the transition probability}
$$

$\Rightarrow$ marginal distribution function of $S$ i.e. $F_S(s)$ is therefore known.

In reality, the market provides us only with a finite number of strikes for each expiry and for each stock $S = S_i, i = 1, \cdots, n$. So how do we fill in Call price functions for each asset for all strikes? Answer related (but only very partially explained) by work on distribution free bounds for one asset, of which we now give a reminder:
A typical Component Option, Procter & Gamble

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The "Market Implied" co-monotonicity gap

- The market only gives us **partial information** about the marginals through the prices of traded options with various traded strikes $K_1^{(i)}, K_2^{(i)}, \ldots, K_{J(i)}^{(i)}$ for stock $S_i$ at a given maturity $t$.

- Let UB be the upper bound for basket option, given only this partial information, then

\[
\text{Market implied comonotonicity Gap} = \text{UB} - \text{traded Market Price}
\]

- **Fundamental**: Given a basket option on $n$ assets, there is a portfolio $\mathcal{P}$ of $n + 1$ options on components, such that

\[
\text{UB} = \text{Market Price of } \mathcal{P}
\]

Below we will discuss how to determine the upper bound UB.
Bertsimas and Popescu, 2003, use a LP approach to derive bounds on assets under a variety of constraints. Here is one of their results:

Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq \ldots \leq K_n$ on a stock $X$, the range of all possible prices for a call option with strike $K$ where $K \in (K_j, K_{j+1})$ for some $j = 0, \ldots, n$ is 

$$[C^-(K), C^+(K)]$$

where

$$C^-(K) = \max \left( C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}}, C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}} \right)$$

lower bounds

$$C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j}$$

upper bounds
Linear interpolation

The interpolated call price function. $\Delta_{j}^{(i)}$ gives the modulus of the slope of $C^{(i)}$ over $(k_{j-1}^{(i)}, k_{j}^{(i)})$.

This graph provides one of many ways of filling in the missing strikes. But it turns out to be the fundamental interpolation, in the case of the upper bound.
The marginals corresponding to piecewise linear call prices are discontinuous at every strike price and constant between strike prices. Because:

\[
\frac{\partial^2 C^{(i)}}{\partial K^2} = \text{density}
\]

and because our call price functions are piecewise linear between two strikes so

\[
\frac{\partial^2 C}{\partial K^2} = 0, \quad K_i^j \leq K \leq K_i^{j+1}
\]

\[
\frac{\partial^2 C}{\partial K^2} = \delta(K_i^j) \times \left(\text{change of slope at } K_i^j\right),
\]

This is illustrated in following slide:
Underlying assets have jumps and regions with no mass

The interpolated call price function. \( \Delta_j^{(i)} \) gives the modulus of the slope of \( \overline{C}^{(i)} \) over \( (k_j^{(i)} - 1, k_j^{(i)}) \).
Now the market implied co-monotonic optimizer \((\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n)\) is a random variable which is distributed like the vector random variable

\[
\left( (F_{S_1}^M)^{-1}(U), (F_{S_2}^M)^{-1}(U), \ldots, (F_{S_n}^M)^{-1}(U) \right)
\]

where \(F_{S_i}^M, i = 1, \ldots, n\) are the market implied marginals with point masses at the strikes.

It can be shown (Laurence and Wang (2004, 2005) and Hobson, Laurence and Wang (2005)) that the market implied co-monotonic optimizer is a solution of optimization problem on next slide:
Constrained optimization problem. Determine

\[
\sup_{\mu} \int \left( \sum_i w_i S_i - K \right)^+ \mu(dS)
\]

subject to

\[
\int (S_i - k_j^{(i)})^+ \mu(dS) = C^{(i)}(k_j^{(i)}), \quad \text{for } i = 1, \ldots, n, j = 1, \ldots, J^{(i)}
\]

\[
\int \mu(dS) = 1
\]
Dual problem

\[
\inf_{\nu, \psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_{j}^{(i)}) \nu_{i}^{j} + \psi
\]

subject to

\[
\left( \sum_{i} w_{i} S_{i} - K \right)^{+} \leq \sum_{i,j} \left( S_{i} - k_{j}^{(i)} \right)^{+} \nu_{i}^{j} + \psi \quad (*)
\]

\( \nu_{j}^{i} \in \mathbb{R}, \text{ for } i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)} \)

\( \psi \in \mathbb{R} \)

\((*)\) is the super-replication condition

Here \( \psi \) is cash component and \( \nu_{j}^{i} \) is number of options with strike \( k_{j}^{i} \) in hedging portfolio.
Finite market - Using all traded options

**Preliminaries** For simplicity of exposition assume all slopes \( \frac{\partial C^{(i)}(u)}{\partial u} \bigg|_{u=k_j^{(i)}} \) are different as \( i \) and \( j \) vary. Let \( I_n = \{1, 2, \cdots, n\} \) where \( n \) is the number of assets.

There is a privileged index \( \hat{i} \in I_n \) such that:

For any model which is consistent with the observed call prices \( C^{(i)}(K_j) \), the price \( B(K) \) for the basket option is bounded above by \( \overline{B_F}(K) \), where

Case I: \( \sum_i w_i k_{J(i)}^{(i)} > K \):

\[
\overline{B_F}(K) = \sum_{i \in I_n \setminus \hat{i}} w_i C^{(i)} \left( k_{J(i)}^{(i)} \right) + w_{\hat{i}} \left\{ (1 - \theta_{\hat{i}}^*) C^{(i)} \left( K_{J(\hat{i})}^{(\hat{i})} - 1 \right) + \theta_{\hat{i}}^* C^{(i)} \left( k_{J(\hat{i})}^{(\hat{i})} \right) \right\}
\]

\( \theta_{\hat{i}}^* \) is defined as

\[
\theta_{\hat{i}}^* = \frac{\bar{x}_{\hat{i}}^*-\bar{x}_{i}^- (\phi^*)}{\bar{x}_{i}^- (\phi^*)-\bar{x}_{\hat{i}}^- (\phi^*)} = \frac{(K \bar{x}_{\hat{i}}^- / w_{\hat{i}}) k_{J(\hat{i})}^{(\hat{i})} - K_{J(\hat{i})}^{(\hat{i})} - 1}{k_{J(\hat{i})}^{(\hat{i})} - k_{J(\hat{i})}^{(\hat{i})} - 1}, \bar{x}_{\hat{i}}^* \in [k_{J(\hat{i})}^{-1}, k_{J(\hat{i})}^-].
\]
Case II: $\sum_i w_i K_{J(i)} \leq K$:

$$\overline{B}_F(K) = \sum_i w_i C^{(i)} s^{(i)}_{k^{(i)}_{J(i)}}$$

Based on experiments with real data, the second case essentially never arises in practice.

Moreover, the upper bound is optimal in the sense that we can find co-monotonic models which are consistent with the observed call prices and for which the arbitrage-free price for the basket option is arbitrarily close to $\overline{B}_F(K)$.

So where’s the beef in Case I?

All the beef in fleshing out the estimate in the first case is in determining the special index $\hat{i}$ and the indices $j(i)$, $i = 1 \cdots n$. 
How to find which options to choose?

Possible to show that there is **No cash component** $\psi$ in the optimal portfolio. So can consider super-replicating portfolios consisting entirely of options with various strikes (some of which may have strike zero).

The upper bound is available in quasi-closed form, meaning there is a simple algorithm to determine the solution, modulo a slope ordering algorithm: Order all slopes of all call price functions and cycle through.

To get the intuition as to how to proceed, note that if $\sum \lambda_i = 1$ then

$$\left( \sum_i w_i X^{(i)}_M - K \right)^+ \leq \sum_i w_i \left( X^{(i)}_M - \frac{\lambda_i K}{w_i} \right)^+,$$

due to Merton

So that

$$C_B(K) \leq \sum_i w_i C^{(i)} \left( \frac{\lambda_i K}{w_i} \right).$$

The $\lambda_i$ are arbitrary and so

$$C_B(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)} \left( \frac{\lambda_i K}{w_i} \right).$$
We wish to find the infimum of $\sum_i w_i C^{(i)}(\lambda_i K/w_i)$ over choices $\lambda_i$ satisfying $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Define the Lagrangian

$$L(\lambda, \phi) = \sum_i w_i C^{(i)}(\lambda_i K/w_i) + \phi \left( \sum_i \lambda_i - 1 \right).$$

Objective function is convex but only $C^{0,1}$, because each piecewise linear call price functions $C^{(i)}$, is $C^{0,1}$, i.e. $\frac{\partial C^{(i)}_i}{\partial K}$ has a jump at each strike $K_{i,j}^j$, $j = 1, \ldots, n_i$.

Note that objective functional is separable function of 1-dimensional functions.

Therefore for each fixed Lagrange Multiplier $\phi$, the gradient can point in a cone of different directions. In the terminology of convex analysis we have $\phi/\beta K \in \partial C^{(i)}(\lambda_i K/w_i)$, where $\partial$ is the subdifferential of the function $C^{(i)}$. \

Illustration Min

\[ \phi + S^- < 0 \]
\[ \phi + S^+ > 0 \]
\[ -S^+ < \phi < -S^- \]

unique min
at \( K_i \)

\( -\phi \) too large \( \rightarrow \) no min or min for smaller \( K \)
For each $\phi$ there is either a unique $\lambda(\phi)$ or an interval $[\lambda^-(\phi), \lambda^+(\phi)]$.

Essentially:
- $[\lambda(\phi)^-, \lambda(\phi)^+] \sim [w_i K_i^j / K, w_i K_i^{j+1} / K]$ for some $i$ and $j$.
- So Algorithm:
  - Order all the slopes of all call price functions. i.e. if 30 assets and 8 non zero strikes, order 240 slopes.

  $S_1 \leq S_2 \leq \cdots \leq S_{240}$

  - Now starting with $\phi = \epsilon << 1$ increase $\phi$ while monitoring the quantity

    $$\Lambda(\phi) = \sum \lambda^+(\phi)$$

    which starts very large for small $\phi$ (\Rightarrow large $K_i^j$) and decreases as $\phi \uparrow$.

  - The first time $\Lambda(\phi)$ crosses 1. STOP! $\Leftrightarrow$ Optimal value of $\phi = \phi^*$ has been reached.
We now illustrate the output on real DJX data.

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**TABLE 4.** The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicating portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In others the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset increases as the strike on the DJX increases.
How good is the Upper Bound? Spot was 99.07

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<th>DJX Strikes</th>
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<th>UB Clean Data</th>
<th>BS Price $\rho = 0$</th>
<th>BS Price $\rho = .5$</th>
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PART II
The methodology for basket options can also be applied to generalized spread options.

The payoff $\psi$ of the generalized spread options

$$\psi(S_1, \cdots, S_n) = \left( \sum_{i=1}^{n} w_i S_i - K \right)^+$$

where the weights $w_i$ are constants of arbitrary sign.

Examples contain heating oil crack spread

$$((42 \times [HO] - [CO] - K)^+)$$

3:2:1 crack spread

$$((42 \times \frac{2}{3}[UG] + 42 \times \frac{1}{3}[HO] - [CO] - K))$$

Note: 1 barrel = 42 gallons
Antimonicotonicity instead

Let us group the payoff function for the generalized spread option as

$$\psi(S_1, \cdots, S_n) = \left( \sum_{i \in I^+} w_i S_i - \sum_{i \in I^-} |w_i| (S_i - K) \right)^+$$

where $I^+$ denotes the set of indices with positive weights and $I^-$ the negative weights.

The upper bound is attained when

- Assets indexed in $I^+$ are comonotonic to one another.
- Assets indexed in $I^-$ are also c-monotonic to one another.
- Any asset in $I^+$ is antimonicotonic to every asset in $I^-$.  

Special case: $\psi(S_1, S_2) = (S_1 - S_2 - K)^+$

Upper bound is attained when $S_1$ and $S_2$ are antimonicotonic.
Recall the definition of anti-monotonicity:

A two dimensional random vector \((X_1, X_2)\) is said to be anti-monotonic if there exists a uniformly distributed random variable \(U\) such that

\[
U \sim \text{Uniform}(0, 1) \\
(X_1, X_2) \overset{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(1 - U)) ,
\]

where \(F_{X_i}(x)\) is the distribution function of \(X_i\).
Therefore, for the generalized spread options with payoff

$$\left( \sum_{i \in I^+} w_i S_i - \sum_{i \in I^-} |w_i| S_i - K \right)^+,$$

the upper bound is attained if there exists a uniformly distributed random variable $U \sim \text{Uniform}(0, 1)$ such that

- $S_i \overset{d}{=} F_{X_i}^{-1}(U)$ for $i \in I^+$
- $S_i \overset{d}{=} F_{X_i}^{-1}(1 - U)$ for $i \in I^-$

where $F_{S_i}(x)$ is the distribution function of $S_i$. 
Super hedge portfolio

- Observe the inequality

\[
\left( \sum_{i \in I^+} w_i S_i - \sum_{i \in I^-} |w_i| S_i - K \right)^+ \leq \sum_{i \in I^+} w_i \left( S_i - \frac{\lambda_i K}{w_i} \right)^+ + \sum_{i \in I^-} |w_i| \left( \frac{\lambda_i K}{|w_i|} - S_i \right)^+
\]

where \( \lambda_i \geq 0 \) and \( \sum_{i \in I^+} \lambda_i - \sum_{i \in I^-} \lambda_i = 1 \).

- Taking expectation on both sides of the inequality we have

\[
\text{Spread option price} \leq \sum_{i \in I^+} w_i C_{S_i} \left( \frac{\lambda_i K}{w_i} \right) + \sum_{i \in I^-} |w_i| P_{S_i} \left( \frac{\lambda_i K}{|w_i|} \right)
\]

where \( C_{S_i}(k) \) and \( P_{S_i}(k) \) are the call and put prices of \( S_i \) struck at \( k \) respectively.

- The super hedge portfolio is therefore obtained by minimizing the right hand side over the constrained parameters \( \lambda_1, \ldots, \lambda_n \).

- The portfolio consists of buying calls for the components with positive weight and puts for components with negative weights.
Optimal solution

- As in the basket case, the constrained minimization problem is solved by the method of Lagrange multipliers.

- Again the slopes $\Delta_j^{(i)}$ are ordered as a (strictly) decreasing sequence $\Delta_1, \cdots, \Delta_N$ with repetitions removed, where

$$\Delta_j^{(i)} = \frac{c_j^{(i)} - c_{j-1}^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^+$$

- Gather together all slopes

$$\Delta_j^{(i)} = \frac{p_j^{(i)} - p_{j-1}^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^-$$

- Puts and calls

- Corresponding to each slope $\Delta_l$, $\lambda_i(l) = \frac{w_i k_j^{(i)} j_i(l)}{K}$ is assigned to asset $i$, where

$$j_i(l) = \max\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^+$$

$$j_i(l) = \min\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^-$$
Optimal solution

Starting with \( l = N \), let us iteratively decrease \( l \) by one, until

\[
\sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1.
\]

Denote the critical \( l \) by \( l^* \). If the condition \( \sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1 \) is not exactly satisfied, linearly interpolate the \( \lambda_i \)'s for those indices \( i \), which change when \( l \) decreases from \( l^* \) to \( l^* - 1 \). Denote the interpolation factor by \( \theta^* \) and these indices by \( I^+_l \) and \( I^-_l \) for positive and negative weights respectively.

Case I: \( \sum_{i \in I^+} w_i k_i^{(i)} > K \) and \( \sum_{i \in I^+} \lambda_i(l^*) - \sum_{i \in I^-} \lambda_i(l^*) = 1 \)

\[
UB = \sum_{i \in I^+} C^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right) + \sum_{i \in I^-} P^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right)
\]
Case II: $\sum_{i \in I^+} w_i k_i^{(i)} > K$ and $\sum_{i \in I^+} \lambda_i(l^*) - \sum_{i \in I^-} \lambda_i(l^*) > 1$

$$UB = \sum_{i \in I^+ \setminus I_{l^*}^+} w_i C^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right) + \sum_{i \in I^- \setminus I_{l^*}^-} w_i P^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right)$$

$$+ \sum_{i \in I_{l^*}^+} w_i \left[ \theta^* C^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right) + (1 - \theta^*) \theta^* C^{(i)} \left( \frac{w_i k_i^{(i)}}{K} - 1 \right) \right]$$

$$+ \sum_{i \in I_{l^*}^-} w_i \left[ \theta^* P^{(i)} \left( \frac{w_i k_i^{(i)}}{K} \right) + (1 - \theta^*) P^{(i)} \left( \frac{w_i k_i^{(i)}}{K} + 1 \right) \right]$$

Case III: $\sum_{i \in I^+} w_i k_i^{(i)} \leq K$,

$$UB = \sum_{i \in I^+} w_i C^{(i)} (k_i^{(i)})$$
Simulation illustration

<table>
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<th>Hedging Price</th>
<th>MC Price</th>
<th>MC accuracy</th>
<th>$S_1$ strike</th>
<th>$C$</th>
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$S_1$ and $S_2$ are distributed like two antimonotonic geometric Brownian motions (equivalently the instantaneous correlation $\rho$ equals $-1$) with parameters $\sigma_1 = .355$, $\sigma_2 = .2$, $T = .5$, $r = 0$, $d_1 = d_2 = 0$. The Monte Carlo prices are computed using $n = 50,000$ paths. The spot prices are $S_1 = 1.48$, $S_2 = 59.33$, and the weights are $w_1 = 42$, $w_2 = 1$. The strikes that were actually trading are given by the NYMEX data for the December 2006 contract.
Empirical analysis

The results of monitoring the crack spread option, difference between heating and crude oil for the contract that expired December 2006 are shown in the following table. The table shows the true price in the third column and the lower and upper bounds in column 2 and 4. The comononotonicity and antimonotonicity gaps are shown next, as well as their relative counterparts.
### Empirical analysis

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A New Tool For Correlation Risk Management: The Market Implied Comonotonicity Gap – p. 45/51
### Empirical analysis

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Figure 1: Time dependence of the comonotonicity gap

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Day by Day
day by day

To see how the gaps can generate a profit, suppose for instance that on October 13th we sell the comonotonicity gap $G^c$ for 1.52 (sell spread option and buy optimal subreplicating portfolio). Then on November 21st, we buy back the gap for 0.1. If the annualized interest rate is 0.05, we have made a profit of 1.51. Also, in our data set, $G^a$ is monotonically decreasing, so we can sell the antimonotonicity gap on October 6th and buy it back for a profit at almost any later date. The data set also appears to indicate some arbitrage opportunities, but this may be offset by bid ask spreads or lack of liquidity.
Conclusions

- We have discussed the market implied comonotonicity gap as a tool for dispersion trading. Here it has been illustrated empirically in the case of spread options.

- Many open problems:
  - Lower bound for basket options for more than two assets
  - Lower bound for two assets and more than one strike constraint.

- Add constraints on the correlation(s).

- Statistical testing needed to determine optimal time to enter into a Gap strategy. Studies of profit and loss over periods of a year or more needed.
Slogan: MIND THE GAP!