On the Wealth Dynamics of Self-financing Portfolios under Endogenous Prices

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Joint work with Jesper Pedersen and Klaus Schenk-Hoppé.
Motivation

**Mathematical Finance**
- Classical continuous time theory
- Price process given
- Option pricing
- Optimal investment

**Economics**
- Supply and demand
- Prices by market clearing
- Interaction of investors

- Evolution of investors’ wealth
- Price formation
- Optimal strategies
Classical continuous-time finance

- Investors are **price-takers**
- Trades have **no impact** on the market
- Dynamics of asset prices are given by a stochastic process, e.g.
  \[ S_t = S_0 \exp(\mu t + \sigma B_t). \]
- There is **infinite supply** of financial assets
- There is **infinite divisibility** of financial assets

Standing assumption

Small investors!!!

- **Infinite divisibility** of financial assets \( \Rightarrow \) **big investors**
Large trader and large trades

1. Option hedging has **significant impact** on stock prices
   - Empirical “proofs”

2. Large trades cannot be performed without being noticed
   - **splitting** large trades into smaller to lower market impact – algorithmic trading
   - using strategies based on econometric and mathematical reasoning: Keym and Madhavan (1996), He and Mamaysky (2005)
   - strategies based on analysis of limit order books

Limitations
- only one large trader
- trader’s impact on the market is ad-hoc specified
Equilibrium with heterogeneous agents

- many investors, heterogeneous beliefs
- dividends
- investors are utility maximizers
- prices determined to clear the market
- one-period models and overlapping generations (De Long, Shleifer, Summers, Waldmann)
- **dynamic models** are very complicated and often unsolvable (Hommes)
### The Market

<table>
<thead>
<tr>
<th>Asset $k$</th>
<th>$k = 1, 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price $S_k(t)$</td>
<td></td>
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<tr>
<td>Cumulative dividends $D_k(t)$</td>
<td></td>
</tr>
</tbody>
</table>

$$D_k(t) = \int_0^t \delta_k(s) ds$$

Assets in net supply of 1.
### The Market

#### Asset $k$

- **$k = 1, 2$**
- **Price $S_k(t)$**
- **Cumulative dividends $D_k(t)$**

\[ D_k(t) = \int_0^t \delta_k(s) ds \]

#### Investor $i$

- **$i = 1, 2$**
- **Wealth $V^i(t)$**
- **Consumption rate $cV^i(t)$**
- **Constant proportions trading strategy $(\lambda^i_1, \lambda^i_2)$**

Assets in net supply of 1.

Portfolio number of shares of asset $k$:

\[ \frac{\lambda^i_k V^i(t)}{S_k(t)} \]
Wealth dynamics

\[ dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption} \]
Wealth dynamics

\[ dV_i(t) = \text{capital gains} + \text{dividends} - \text{consumption} \]

Capital gains

\[ \sum_{k=1}^{2} \lambda_k \frac{V_i(t)}{S_k(t)} dS_k(t) \]
Wealth dynamics

\[ dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption} \]

Capital gains

\[ \sum_{k=1}^{2} \frac{\lambda_k^i V^i(t)}{S_k(t)} dS_k(t) \]

Dividends

\[ \sum_{k=1}^{2} \frac{\lambda_k^i V^i(t)}{S_k(t)} dD_k(t) \]
Wealth dynamics

\[ dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption} \]

Capital gains

\[ \sum_{k=1}^{2} \frac{\lambda_k}{S_k(t)} V^i(t) dS_k(t) \]

Dividends

\[ \sum_{k=1}^{2} \frac{\lambda_k}{S_k(t)} V^i(t) dD_k(t) \]

Consumption

\[ cV^i(t) dt \]
Wealth dynamics

\[ dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption} \]

\[ dV^i(t) = \sum_{k=1}^{2} \frac{\lambda_k^i V^i(t)}{S_k(t)} \left( dS_k(t) + dD_k(t) \right) - cV^i(t)dt \]
Market clearing condition

\[ \frac{\lambda_1^1 V^i(t)}{S_k(t)} + \frac{\lambda_2^2 V^i(t)}{S_k(t)} = 1, \quad k = 1, 2. \]

Equivalent to the net clearing condition:

\[ d\theta^1_k(t) + d\theta^2_k(t) = 0, \quad k = 1, 2. \]
Price formation

Dividend intensities $\delta_k(t)$

+ 

Investment strategies $(\lambda_1^i, \lambda_2^i)$

+ 

Investor’s wealth dynamics

+ 

Market clearing condition

⇓

Asset prices $S_k(t), \ k = 1, 2$
Theorem

1. For any feasible \((V^1(0), V^2(0))\) there exists a unique \((V^1(t), V^2(t))\) satisfying wealth dynamics and market clearing condition.

2. Asset price dynamics are given by

\[
S_k(t) = \lambda^1_k V^1(t) + \lambda^2_k V^2(t), \quad k = 1, 2.
\]
Markovian dividend intensities

Relative dividend intensity

\[ \rho(t) = \frac{\delta_1(t)}{\delta_1(t) + \delta_2(t)} \in [0, 1] \]

Assumptions

1. \( \rho(t) \) is a positively recurrent Markov process
2. Its state space is countable
3. Its initial distribution is stationary (stationary economy)

Theorem

Relative dividend intensity process is ergodic:

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \rho(s) \, ds = E\rho(0). \]
Theorem

If investor 1 follows strategy

\[ \Pi^* = (\lambda_1^1, \lambda_2^1) = (\mathbb{E}\rho(0), 1 - \mathbb{E}\rho(0)) \]

and investor 2 follows a strategy \((\lambda_1^2, \lambda_2^2) \neq \Pi^*\) then

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{V^1(s)}{V^1(s) + V^2(s)} \, ds = 1.
\]

Remarks

1. \(\Pi^*\) is based on fundamental valuation.
2. Relative wealth of investor 2 converges to zero.
3. At odds with findings in discrete-time evolutionary models (Evstigneev, Hens, Schenk-Hoppé).
Price dynamics

If one of the investors follows trading strategy $\Pi^*$ then asset prices converge:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{S_1(s)}{S_1(s) + S_2(s)} ds = \mathbb{E}\rho(0).$$
Price dynamics

If one of the investors follows trading strategy $\Pi^*$ then asset prices converge:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{S_1(s)}{S_1(s) + S_2(s)} ds = E\rho(0).$$

**Fundamental valuation**

$$\frac{E\delta_1(0)}{E\delta_1(0) + E\delta_2(0)}$$

**Our valuation**

$$E\left(\frac{\delta_1(0)}{\delta_1(0) + \delta_2(0)}\right)$$

**Remarks**

1. Fundamental valuation comes as a result of computing average historical payoffs.
2. Our valuation is a fundamentally different benchmark.
Almost sure convergence

**Assumption**
For every state $x$
\[ \mathbb{E}^x(\tau_x)^2 < \infty. \]

**Theorem**
1. *If investor 1 follows strategy $\Pi^*$ and investor 2 follows a strategy $(\lambda_1^2, \lambda_2^2) \neq \Pi^*$ then*

\[
\lim_{t \to \infty} \frac{V_1(t)}{V_1(t) + V_2(t)} = 1 \quad \text{a.s.}
\]

2. *If one of the investors follows strategy $\Pi^*$ then asset prices converge to our benchmark value:*

\[
\lim_{t \to \infty} \frac{S_1(t)}{S_1(t) + S_2(t)} = \mathbb{E}\rho(0) \quad \text{a.s.}
\]
### Proof

**What we hoped to do**

- Linearization and Lagrange multipliers
- Multiplicative Ergodic Theorem

**Why?** It works fine in discrete-time.

- Continuous-time setting *suprised us*. Lagrange multiplier at the steady state is *zero!*

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**What we have done**

- Domination by a Ricatti-type equation with random coefficients.
- One coefficient depending on the solution of the original problem.
- Arcsine law.
Summary

- Heterogeneous investors in continuous time model
- Wealth dynamics
- Optimal investment strategies
- Asset pricing - new valuation benchmark

Open problems
- Time varying investment strategies
- More agents