Optimal Dividends in Presence of Downside Risk

(joint work with Luis H. R. Alvarez E.)

Teppo A. Rakkolainen
Turku School of Economics

teppo.rakkolainen@tse.fi

AMaMeF Mid-Term Conference in Vienna, September 20, 2007
Outline

1. Introduction
2. Basic Assumptions and Setup
3. Some Auxiliary Results
4. Main Theorem: Optimal Singular Control of Dividends
5. References
Dividend Payout Problem

- **Economic problem**: in what way should a firm pay out dividends in order to maximize the expected present value of future dividends to shareholders?
- **Mathematical problem**: to determine the optimal control policy for a stochastically fluctuating process.
- Answers or at least partial answers are well known in cases when the underlying process is a *linear diffusion* or a *Lévy process* (arithmetic or exponential).
- But what about other jump diffusions?
Downside Risk

- We consider *spectrally negative* jump diffusions, i.e. processes which have only downward jumps but increase continuously. These downward discontinuities represent the *downside risk*.

- Motivation for this model is twofold:
  - The markets tend to react to bad news more dramatically than to good news.
  - Principle of prudence: it is prudent to take into account the potential adverse events (say, instantaneous drops in asset value) and disregard uncertain future profits.
Our Goal

- We shall state reasonably general sufficient conditions for the optimal singular stochastic dividend control to be a barrier strategy (except for a potential initial lump sum dividend at time 0).
- We will extend the representation of the value function in terms of the minimal increasing $r$-excessive map (known in linear diffusion case) to our setup.
- This result implies similar results and representations for the associated optimal impulse control (optimality of a target–trigger policy) and optimal stopping problems (optimality of a single threshold rule).
Underlying Lévy Diffusion $X$

- The reservoir of assets from which dividends are paid out evolves on $I := (0, \infty)$ according to

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - \int_{(0,1)} X_t z \tilde{N}(dt, dz),$$  \hspace{1cm} (1)

$X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated Poisson point process with characteristic measure $\nu = \lambda m$, and jump size distribution $m$ has a continuous density.

- $\mu \in C^1$ and $\sigma > 0$ are assumed to satisfy the usual conditions for the existence of a strong solution.
Assumptions on $X$

- The absence of speculative bubbles condition

\[
\mathbb{E}_x \int_0^\infty e^{-rs} X_s ds < \infty,
\]

where $r > 0$ is the discount rate, is met.

- The boundaries 0 and $\infty$ are natural for $X$, i.e. unattainable in finite time.

- $X$ is regular in the sense that for all $x, y \in I$ it holds that $\mathbb{P}_x(\tau_y < \infty) = 1$, where $\tau_y = \inf\{ t > 0 : X_t \geq y \}$. 

Infinitesimal Generator of $X$

- Operator coinciding with the infinitesimal generator of $X$ is defined for $f$ sufficiently smooth by

$$ (Gf)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) + \lambda \int_{(0,1)} \{f(x - xz) - f(x) + xzf'(x)\}m(dz). $$

- We assume that there exists an increasing $C^2$ solution $\psi$ of $G_r\psi := G\psi - r\psi = 0$ such that $\psi(0) = 0$. 

Optimal Dividends in Presence of Downside Risk

Basic Assumptions and Setup

Teppo Rakkolainen
Associated Continuous Diffusion $\tilde{X}$

We define an associated diffusion $\tilde{X}$ by

$$d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dW_t,$$

where $\tilde{\mu}(x) = \mu(x) + \lambda x \cdot \int_{(0,1)} z m(dz) = \mu(x) + \lambda \tilde{z} x$. 

Optimal Dividends in Presence of Downside Risk

Basic Assumptions and Setup
Controlled Dynamics

- The controlled cash flow dynamics $X_t^D$ are characterized by the stochastic differential equation

$$dX_t^D = \mu(X_t^D)dt + \sigma(X_t^D)dW_t - \int_{(0,1)} X_t^D z \tilde{N}(dt, dz) - dD_t,$$

$X_0^D = x$, where $D$ denotes the implemented dividend policy.

- A dividend payout strategy is *admissible* if it is non-negative, adapted, cádlág, and non-decreasing; the class of admissible policies is denoted by $\mathcal{A}$. 
Cash Flow Management Problem

- Objective is to solve the singular stochastic control problem

\[
V_S(x) = \sup_{D \in A} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s, \tag{6}
\]

where \( \tau_0^D = \inf\{ t > 0 : X_t^D \leq 0 \} \) denotes the lifetime of \( X^D \).

- It is worth emphasizing that in our model liquidation is always the result of a control action (and, thus, endogenous), as the assumed boundary behavior of \( X \) implies that exogenous liquidation in finite time is not possible.
Net Appreciation Rate

- Define the net appreciation rate $\rho : I \rightarrow \mathbb{R}$ of the stock $X$ as $\rho(x) = \mu(x) - rx$ and assume throughout that it has a finite expected cumulative present value.

- This mapping plays a key role in the determination of the optimal payout policy and its value.
Auxiliary Mappings

- define the $C^1$ mappings $H : l^2 \rightarrow \mathbb{R}$ as

$$H(x, y) = \begin{cases} 
x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\
\frac{\psi(x)}{\psi(y)} & x < y.
\end{cases} \quad (7)$$

- For a given fixed $y \in l$ the function $x \mapsto H(x, y)$ satisfies the variational equalities

$$(G_r H)(x, y) = 0, \quad x < y$$

$$\partial_x H(x, y) = 1, \quad x \geq y.$$
A Crucial Uniqueness and Existence Result (Theorem 1)

Theorem

Assume that the net appreciation rate $\rho(x)$ satisfies the limiting inequalities $\lim_{x \to \infty} \rho(x) < 0 \leq \lim_{x \downarrow 0} \rho(x)$, that there exists a unique threshold $\hat{x} \in I$ such that $\rho(x)$ is increasing on $(0, \hat{x})$ and decreasing on $(\hat{x}, \infty)$, and that $\rho(x)$ is concave on $(\hat{x}, \infty)$. Then equation $\psi''(x) = 0$ has a unique root $x^* \in (\hat{x}, \infty)$ so that $\psi''(x) \geq 0$ for $x \leq x^*$ and $x^* = \arg\min\{\psi'(x)\}$. 
Sketch of Proof (Existence)

- To prove existence, first establish local concavity of $\psi(x)$ near the origin, then show that it cannot become convex on $(0, \hat{x})$ and finally that it has to become convex before $x_0 = \rho^{-1}(0)$.

- To do this by contradiction, use the auxiliary quantity

$$I(x) = r(\psi(x) - x\psi'(x)) - \rho(x)\psi'(x) - J(x, \psi(x)), \quad (8)$$

where $I(x) = \frac{1}{2}\sigma^2(x)\psi''(x)$, and

$$J(x, \psi(x)) = \int_{(0,1)} \{\psi(x - xz) - \psi(x) + xz\psi'(x)\} \nu(dz). \quad (9)$$
Sketch of Proof (Uniqueness)

To establish uniqueness, consider the derivative of

\[ \tilde{I}(x) = (r + \lambda) \left( \frac{\psi(x)}{S'(x)} - x \frac{\psi'(x)}{S'(x)} \right) - \tilde{\rho}(x) \frac{\psi'(x)}{S'(x)} - \tilde{J}(x), \quad (10) \]

where \( \tilde{I}(x) = \frac{\sigma^2(x)\psi''(x)}{2S'(x)} \), \( \tilde{\rho}(x) = \rho(x) - \lambda x (1 - \tilde{z}) \), \( S'(x) = \exp \left( - \int \frac{2\tilde{\mu}(x)dx}{\sigma^2(x)} \right) \) denotes the scale density of the associated diffusion \( \tilde{X} \), and

\[ \tilde{J}(x) = \int_{(0,1)} \frac{\psi(x(1-z))}{S'(x)} \nu(dz). \]

Using concavity of \( \rho(x) \), the fact that \( \tilde{I}'(x^*) > 0 \) and Leibniz rule, show that once positive, \( \tilde{I}'(x) \) cannot turn negative on \( (x^*, \infty) \).
A Superharmonicity Theorem

Theorem

Suppose that the assumptions of Theorem 1 are satisfied and define the function $F : I \mapsto \mathbb{R}_+$ as $F(x) = H(x, x^*)$. Then,

(A) $F \in C^2(I)$, $(G_r F)(x) \leq 0$, $F'(x) \geq 1$, and $F''(x) \leq 0$ for all $x \in I$, and

(B) $F(x) \geq H(x, y)$ and $F'(x) \geq H_x(x, y)$ for all $x, y \in I^2$ and $H_y(x, y) < 0$ for all $(x, y) \in \mathbb{R}_+ \times (x^*, \infty)$. 
Sketch of Proof

- (A): use properties of $G_r F(x)$ and its derivative together with the strict concavity of $\psi(x)$ on $(0, x^*)$.
- (B) follows from known results by Alvarez and Virtanen.
Optimal Singualr Control of Dividends

Theorem

Assume that the assumptions of Theorem 1 are satisfied. Then the value of the singular control problem is given by \( V_S(x) = H(x, x^*) \). The value is twice continuously differentiable, monotonically increasing and concave. Moreover, the marginal value (Tobin’s marginal q) of the singular control reads as

\[
V'_S(x) = \psi'(x) \sup_{y \geq x} \left\{ \frac{1}{\psi'(y)} \right\} = \begin{cases} 
1 & x \geq x^* \\
\frac{\psi'(x)}{\psi'(x^*)} & x < x^*.
\end{cases} 
\] (11)

The corresponding optimal singular control consists of an initial impulse \( \xi_{0-} = (x - x^*)^+ \) and a barrier strategy where retained earnings in excess of \( x^* \) are instantaneously paid out as dividends.
Sketch of Proof

- Take any \( D \in \mathcal{A} \), apply the generalized Itô formula to \((t, x) \mapsto e^{-rt} H(x, x^*)\) for a suitable sequence of increasing stopping times and use the superharmonicity theorem and monotone convergence to establish that the proposed value function dominates the value obtained by strategy \( D \).
- Show that the proposed strategy is admissible.
Further Results

- It can be shown that the obtained representation of the value of the singular control problem implies that also the associated impulse control problem

\[
V^c_I(x) = \sup_{(\tau, \xi) \in \mathcal{V}} J^{\tau, \xi}(x) = E_x \left[ \sum_{i=1}^{N} e^{-r\hat{\tau}(i)} (\hat{\xi}(i) - c) \right]
\]

as well as the associated optimal stopping problems

\[
V_{OSP}(x) = \sup_{\tau \in \mathcal{T}} E_x \left[ e^{-r\tau} X_{\tau} \right] \tag{12}
\]

and

\[
V^c_{OSP}(x) = \sup_{\tau \in \mathcal{T}} E_x \left[ e^{-r\tau} (X_{\tau} - c) \right], \tag{13}
\]

where \( \mathcal{T} \) is the set of all \( \mathbb{F} \)-stopping times, are solvable in terms of the minimal increasing \( r \)-excessive map.
References

